HILBERT SPACES OF ANALYTIC FUNCTIONS

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DEDICATED

TO THE

MEMORY

OF MY

MOTHER

CERTIFICATE

This is to certify that the work embodied in the thesis "Hilbert Spaces of Analytic Functions" by Mr. G.D. Lakhani has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

R. S. L. Srivastava
(R.S.L. SRIVASTAVA) 28/4/71

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CONTENTS

CHAPTER I

INTRODUCTION

The study of square summable power series, concerning the extinction and closure problems for linear transformations in Hilbert spaces originated with the work of A. Beurling [2] . He constructed a model of a linear transformation T defined in a Hilbert space H such that $T^n \longrightarrow 0$ as $n \rightarrow \infty$, the adjoint T^* is isometric and the eigenvectors form a fundamental set in H, and found T to be unitarily equivalent to the transformation $f(z) \longrightarrow \dfrac{f(z) - f(0)}{z}$ in a Hilbert space of analytic functions, square integrable on the unit circle. He characterized that the invariant subspaces under the multiplication by z are of the form $B(z)\,H^2$, where B(z) is an analytic function in the disc such that $|B(e^{i\theta})| = 1$, almost everywhere on the boundary, which led to the factorization of B(z) in terms of inner and outer functions ([15]; Chapter 5). This work was extended to vector-valued analytic functions by Helson and Lowdenslager [13] and an analogous factorization for operator-valued analytic functions was given by Lax [17]. This theory also led some workers (de Branges, Rovnyak, Halmos, etc.) to study shift operators and their invariant subspaces, and motivated Sz.-Nagy and Foias to study contraction transformations [24]. Later, the theory was developed to solve many problems of operator theory ([9] , [10]).

1.1 Notations and some basic definitions

The following conventions and notations are followed throughout the thesis. The letter S denotes a fixed complex Hilbert space which is used as a coefficient space. In general, the cardinality of the complete orthonormal base of S is assumed to be infinite and is mentioned explicitely whenever it is finite, (in particular, S is complex field if its dimension is 1). By a vector, or constant, we mean an element of S. $|\ |$ is used for the norm in S. If c is a constant then $\bar{c}a = \langle a,c \rangle$ for every vector a. By an operator we understand a bounded linear transformation defined over S into S. The word operator is reserved for transformations in S. The adjoint of an operator A is denoted by $\bar{A}$. Small letters are used for vector-valued functions whereas capitals are taken to denote operator-valued functions (a function, the range of which is contained in the algebra of operators). By a transformation T we mean a bounded linear transformation in some Hilbert space with its adjoint denoted by T^*. We take F(z) to denote a function whereas its value at some point z=w is denoted by F(w). For definitions of technical terms we refer to the book of Kato [16].

Let D be a domain in the complex plane. A function f(z) defined on D into S is said to be (vector-valued) analytic at z if $\bar{c}f(z)$ is (complex) analytic in some neighbourhood of z for every constant c. A function F(z) defined on D into operator algebra is said to be an (operator-valued) analytic function at z if $\bar{y}F(z)x$ is a (complex) analytic at z for every pair of vectors x,y. These are weaker definitions of analyticity but they imply analyticity in norm topology ([14]; Theorem 3.10.1). Beside these definitions we shall need Phragmén-Lindelöf principle for operator-valued analytic functions

which can be obtained from Theorem 3.13.6 ([14]). We also need uniform convergence of sequence of analytic functions ([14]; Theorem 3.14.1). For general theory of vector-valued analytic functions we refer to the book of Hille and Phillips [14].

1.2 Some basic spaces

Consider the collection of power series $f(z) = \Sigma a_n z^n$ with a_n in S for $n=0,1,2,\ldots$ Define $f(z) + g(z) = \Sigma (a_n + b_n) z^n$ and $\alpha f(z) = \Sigma \alpha a_n z^n$, where $g(z) = \Sigma b_n z^n$ and α is a scalar. Denote by $S(z)$ the set of power series $f(z) = \Sigma a_n z^n$ such that $\| f \|^2 = \Sigma | a_n |^2 < \infty$. Then $S(z)$ is a Hilbert space in the above norm and contains $\frac{f(z) - f(0)}{z}$ whenever $f(z)$ belongs to $S(z)$. Moreover, it has following property for every element $f(z)$.

$$(1.2.1) \qquad \left\| \frac{f(z) - f(0)}{z} \right\|^2 = \| f(z) \|^2 - | f(0) |^2 .$$

The adjoint of the transformation $f(z) \to \frac{f(z) - f(0)}{z}$ in $S(z)$ is an isometry and is given by $f(z) \to zf(z)$. The series $\frac{c}{1-z\overline{w}}$ belongs to $S(z)$ whenever c is a constant and $|w| < 1$. Moreover, the identity

$$\overline{c}\, f(w) = \langle f(z), \frac{c}{1-z\overline{w}} \rangle$$

holds for every element $f(z)$ of $S(z)$ and constant c in S when $|w| < 1$. This implies that power series $f(z)$ converges uniformly on every compact set in $|w| < 1$, hence represents an analytic function in the unit disc. It is a wellknown result that

$$\| f(z) \|^2 = \lim_{r \to 1-} \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^2 d\theta ,$$

which shows that $S(z)$ coincides isometrically with H^2([15]; Cahpter 7).

(An extension of the closure problem [2] for the vector-valued case was studied by Rovnyak [19] and Halmos [11]). An ideal in $S(z)$ is a closed subspace M of $S(z)$ which contains $zf(z)$ whenever it contains $f(z)$. Let $B(z) = \Sigma B_n z^n$ be a power series with operator coefficients such that

$$(1.2.2) \qquad \| B(z)\, c \|^2 = \Sigma | B_n c |^2 = | c |^2$$

whenever c is a constant orthogonal to the intersection of kernels of B_n, $n \geq 0$. Define $B(z)f(z) = \sum_0^\infty \left(\sum_0^n B_{n-k}\, a_k \right) z^n$. Then $g(z) = B(z)f(z)$ belongs to $S(z)$ whenever $f(z)$ belongs to $S(z)$ and $\| g(z) \| = \| f(z) \|$. Denote by $M(B)$ the range of multiplication by $B(z)$ (which is a partially isometric transformation) in $S(z)$. By definition, $B(z)$ converges to an operator-valued analytic function bounded by 1 in the unit disc and assumes isometric values a.e. on the boundary of the unit disc. It was characterized [19] that every ideal M is equal isometrically to $M(B)$ for some power series $B(z)$ with operator coefficients which satisfies (1.2.2).

It is shown in ([5], [11]) that if T is a transformation bounded by 1 defined over $S(z)$ into $S(z)$ such that $T\, zf(z) = z\, Tf(z)$ for every $f(z)$ in $S(z)$, then $T: f(z) \longrightarrow B(z)f(z)$ where $B(z)$ is a power series with operator coefficients which converges to a function bounded by 1 in the unit disc. There are several characterizations of such power series $B(z)$ in terms of its coefficients; one of them is given in [10] . It known that if $B(z)$ is a power series with operator coefficients which converges to a function bounded by 1 in the unit disc then $B(z)f(z)$

belongs to $\mathcal{S}(z)$ and $\|B(z)f(z)\| \leq \|f(z)\|$ for every element $f(z)$ of $\mathcal{S}(z)$. If $f(z)$ is in $\mathcal{S}(z)$, let its B-norm be defined by

$$(1.2.3) \qquad \|f(z)\|_B^2 = \sup \left(\|f(z)+B(z)g(z)\|^2 - \|g(z)\|^2 \right) ,$$

where the supremum is taken over all elements $g(z)$ of $\mathcal{S}(z)$. By $H(B)$ we mean the set of all power series in $\mathcal{S}(z)$ which have finite B-norm. Then $H(B)$ is a Hilbert space in the B-norm and is the orthogonal complement of $\mathcal{M}(B)$ whenever the multiplication by $B(z)$ in $\mathcal{S}(z)$ is a partial isometry. Note that $\|f(z)\|_B \geq \|f(z)\|$. If $f(z)$ is in $H(B)$ then $\frac{f(z)-f(0)}{z}$ belongs to $H(B)$ and

$$(1.2.4) \qquad \left\| \frac{f(z)-f(0)}{z} \right\|_B^2 \leq \|f(z)\|_B^2 - |f(0)|^2 .$$

The adjoint of the transformation $f(z) \to \frac{f(z)-f(0)}{z}$ in $H(B)$ (now onward $R(0)$ will be used to denote this transformation in $H(B)$ space) is given by

$$(1.2.5) \qquad R(0)^*: f(z) \to zf(z) - B(z)\,\tilde{f}(0),$$

where $\tilde{f}(0)$ is determined from the identity

$$\bar{c}\,\tilde{f}(0) = \left\langle f(z) , \frac{B(z)-B(0)}{z} c \right\rangle_B$$

which holds for every constant c. The transformation

$$R(w): f(z) \to \frac{f(z) - f(w)}{z - w}$$

is defined for each element $f(z)$ of $H(B)$ when $|w| < 1$. In fact, if $\frac{f(z) - f(0)}{z}$ is some element of $H(B)$ and if we denote by $f_1(z) = \frac{f(z)-f(0)}{z}$ and $f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z}$ $n=1,2,\ldots,$ then the series

$$f_1(z) + wf_2(z) + w^2f_3(z) + \ldots.$$

converges in the metric of $H(B)$ whenever $|w| < 1$ and its sum is $\frac{f(z) - f(w)}{z - w}$. Therefore, $R(w) = (1-wR(0))^{-1}R(0)$. Its adjoint transformation is

$$R(w)^* : f(z) \to \frac{1}{1-z\overline{w}} (zf(z) - B(z)\, \tilde{f}\, (\overline{w})),$$

where

(1.2.6) $$\overline{c}\, \tilde{f}(\overline{w}) = \langle f(z), \frac{B(z) - B(w)}{z - w} c\rangle_B$$

for all constants c.

As we see that $R(0)$ plays an important role in the factorization theory for operator-valued analytic functions and the study of contraction transformations, so it is desirable to study it thoroughly. Factorization theorems are direct consequences

of existence of invariant subspaces of $R(0)$ in $H(B)$ ([6]; Theorems 3,4). Let $H(A)$ and $H(B)$ be two given spaces. A sufficient condition that $H(A)$ is contained in $H(B)$ such that the inclusion does not increase norms, i.e., $\|f(z)\|_B \le \|f(z)\|_A$ for every $f(z)$ of $H(A)$, is that $B(z)=A(z)C(z)$ such that the space $H(C)$ exists. In this case, $A(z)\,f(z)$ belongs to $H(B)$ for each $f(z)$ of $H(C)$ and $\|A(z)f(z)\|_B \le \|f(z)\|_C$. It can be used to calculate A-norms of the elements of $H(A)$, i.e.,

$$\|f(z)\|_A^2 = \sup\left(\|f(z)+A(z)g(z)\|_B^2 - \|g(z)\|_C^2\right),$$

where the supremum is taken over all $g(z)$ of $H(C)$. Every element $h(z)$ of $H(B)$ has a unique minimal decomposition, $h(z) = f(z) + A(z)g(z)$ with $f(z)$ in $H(A)$, $g(z)$ in $H(C)$ and

$$(1.2.7) \qquad \|h(z)\|_B^2 = \|f(z)\|_A^2 + \|g(z)\|_C^2 .$$

If $h_k(z) = f_k(z) + A(z)g_k(z)$ is a decomposition of $h_k(z)$ in $H(B)$ with $f_k(z)$ in $H(A)$ and $g_k(z)$ in $H(C)$, $k=1,2$ and if atleast one decomposition is minimal, then

$$(1.2.8) \qquad \langle h_1(z), h_2(z)\rangle_B = \langle f_1(z), f_2(z)\rangle_A + \langle g_1(z), g_2(z)\rangle_C .$$

This result leads to the characterization: A necessary and sufficient condition that $H(A)$ be contained isometrically in $H(B)$ is that there is no non-zero element $A(z)g(z)$ in $H(A)$ with $g(z)$ in $H(C)$. An immediate use of (1.2.8) is seen in the calculations of reproducing kernels for $H(B)$. If c is in S and $|w| < 1$ then $\frac{1-B(z)\overline{B}(w)}{1-z\overline{w}}\, c$ belongs to $H(B)$ and

$$(1.2.9) \qquad \bar{c}f(w) = \langle f(z), \frac{1-B(z)\bar{B}(w)}{1-z\bar{w}} c \rangle_B$$

for every element $f(z)$ of $H(B)$.

The study of $R(0)$ in $H(B)$ is related with the study of the transformation $f(z) \to \frac{f(z) - f(0)}{z}$ in $H(B^*)$, where $B^*(z) = \Sigma \bar{B}_n z^n$, provided $B(z) = \Sigma B_n z^n$. In general, equality is not attained in (1.2.4) for every element of $H(B)$. A necessary and sufficient condition that equality is attained in (1.2.4) for every element of $H(B)$, is that $H(B)$ does not contain any non-zero element of the form $B(z)c$ for any non-zero c in S. To over come this difficulty, $H(B)$ is imbedded into an extension space $\mathbb{D}(B)$. By $\mathbb{D}(B)$ we mean a Hilbert space whose elements are pairs of power series. A pair $(f(z), g(z))$ belongs to $\mathbb{D}(B)$ if $f(z)$ belongs to $H(B)$ and if $g(z) = \Sigma a_n z^n$ where

$$z^n f(z) - B(z)\,(a_0 z^{n-1} + \ldots + a_{n-1})$$

belongs to $H(B)$ for every $n = 1, 2, \ldots$, and if the sequence

$$\| z^n f(z) - B(z)\,(a_0 z^{n-1} + \ldots + a_{n-1}) \|_B^2 + |a_0|^2 + \ldots + |a_{n-1}|^2$$

is bounded. The sequence is non-decreasing. Its limit is taken to define the $\| (f(z), g(z)) \|^2_{\mathbb{D}(B)}$. The transformation $(f(z), g(z)) \to f(z)$ is a partial isometry onto $H(B)$ whereas $f(z) \to (f(z), \tilde{f}(z))$ takes $H(B)$ isometrically in $\mathbb{D}(B)$, where $\tilde{f}(z)$ is defined by (1.2.6).

It follows by the definition that the transformation

$$(1.2.10) \qquad (f(z),g(z)) \to \left(\frac{f(z)-f(0)}{z} , zg(z)-B^*(z)f(0) \right)$$

maps $\mathbb{D}(B)$ into $\mathbb{D}(B)$ and the identity

$$\left\| \left(\frac{f(z)-f(0)}{z} , zg(z)-B^*(z)f(0)\right) \right\|^2_{\mathbb{D}(B)} = \| (f(z),g(z)) \|^2_{\mathbb{D}(B)} - |f(0)|^2$$

is satisfied for every $(f(z),g(z))$ of $\mathbb{D}(B)$. This identity corresponds to the inequality given by (1.2.4) for the elements of $H(B)$. The transformation $(f(z),g(z)) \to (g(z),f(z))$ takes $\mathbb{D}(B)$ isometrically onto $\mathbb{D}(B^*)$. An immediate application of $\mathbb{D}(B)$ space is seen in Theorem 12, ([6] ; p. 354) which states that the multiplication by $B^*(z)$ in $S(z)$ is isometric (consequently, $H(B^*)$ is contained isometrically in $S(z)$) if $\|R(0)^{*n} f(z)\|_B \to 0$ as $n \to \infty$. The transformation $f(z) \to \tilde{f}(z)$ (defined by (1.2.6)) on $H(B)$ into $H(B^*)$ is bounded by 1. It is an isometry, if and only if, $H(B^*)$ does not contain any ~~non-zero~~ element of the form $B^*(z)c$ for any non-zero constant c. This was the statement of Theorem 13 ([6] ;p. 354). Therefore, in general, it is possible that equality is attained in (1.2.4) for all elements of $H(B^*)$ but not for all elements of $H(B)$, or the otherway round.

Now we quote some results concerning characterizations of $H(B)$ spaces. Firstly, we give the canonical model of $R(0)$ in $H(B)$ from Theorem 1 ([6] ; p. 347). Let T be a transformation on a Hilbert space H into itself which is bounded by 1, such that the dimension

of the closure of the range of $(1-T^*T)$ does not exceed the dimension of the coefficient space $\mathcal{S}$. If there is no non-zero element f in H such that $\|T^n f\| = \|f\|$ for every n=1,2,..., then T is unitarily equivalent to the transformation $f(z) \to \frac{f(z)-f(0)}{z}$ in some Hilbert space H_0 of formal power series with vector coefficients such that $\frac{f(z)-f(0)}{z}$ belongs to H_0 whenever f(z) belongs to H_0 and

$$\left\| \frac{f(z) - f(0)}{z} \right\|_0^2 = \| f(z)\|_0^2 - |f(0)|^2 .$$

Theorem 11, ([9] ; p.171) further,states that if $\mathcal{S}$ is infinite dimensional and H_0 is a Hilbert space of power series with vector coefficients such that it contains $\frac{f(z)-f(0)}{z}$ whenever f(z) belongs to H_0, and

$$\left\| \frac{f(z)-f(0)}{z} \right\|_0^2 \leq \| f(z) \|_0^2 - |f(0)|^2 \quad ,$$

for all elements f(z) of H_0, then H_0 is equal isometrically to some H(B).

1.3 For the space H(B) to be contained isometrically in $\mathcal{S}(z)$,it is necessary and sufficient (a consequence of (1.2.8))that it contains no non-zero element of the form B(z) l(z) with l(z) in $\mathcal{S}(z)$. Therefore, it requires an estimation of elements of H(B) of this form. The overlapping space $\mathbb{L} = \mathbb{L}_B$ of the space H(B) is the set of elements l(z) of $\mathcal{S}(z)$ such that B(z)l(z) belongs to H(B). The space $\mathbb{L}$ is a Hilbert space in the norm

$$\| l(z)\|_{\mathbb{L}}^2 = \|B(z)l(z)\|_B^2 + \| l(z)\|^2 .$$

Overlapping spaces are examples of $\mathbb{L}(\phi)$ spaces. If $\phi(z)$ is a power series with operator coefficients which converges in the unit disc such that

$$\operatorname{Re} \phi(w) = \tfrac{1}{2}(\phi(w)+\overline{\phi}(w)) \geq 0$$

for $|w| < 1$, then a space $H(B)$ exists (which is called corresponding space), where

$$B(z) = \frac{1-\phi(z)}{1+\phi(z)} .$$

Let $\mathbb{L}(\phi)$ be the set of power series $l(z)$ with vector coefficients such that $(1+B(z))l(z)$ belongs to $H(B)$. Then $\mathbb{L}(\phi)$ is a Hilbert space in the norm

$$\|l(z)\|_{\mathbb{L}(\phi)} = \|(1+B(z))l(z)\|_B .$$

The transformation $l(z) \to \dfrac{l(z)-l(0)}{z}$ is defined over $\mathbb{L}(\phi)$ into $\mathbb{L}(\phi)$ and has isometric adjoint. The elements of $\mathbb{L}(\phi)$ are convergent power series in the unit disc. The series $\frac{1}{2}\,\dfrac{\phi(z)+\overline{\phi}(w)}{1-z\overline{w}}\,c$ belongs to $\mathbb{L}(\phi)$ whenever c is in S and $|w| < 1$, and the identity

$$\overline{c}\, l(w) = \left\langle l(z), \frac{1}{2}\,\frac{\phi(z)+\overline{\phi}(w)}{1-z\overline{w}}\,c \right\rangle_{\mathbb{L}(\phi)}$$

holds for all elements $l(z)$ of $\mathbb{L}(\phi)$. The result of Theorem 1([7];p.125) states that a space $\mathbb{L}(\phi)$ is overlapping space of some $H(B)$, if and only if, $\mathbb{L}(1-\phi)$ exists and the polynomials which belong to $\mathbb{L}(1-\phi)$, form a dense set in $\mathbb{L}(1-\phi)$.

Extension spaces $\mathbb{E}(\emptyset)$ of $\mathbb{L}(\emptyset)$ are introduced in Theorem 10 ([6] ; p. 374). The transformation in $\mathbb{E}(\emptyset)$ corresponding to $l(z) \to \frac{l(z)-l(0)}{z}$ in $\mathbb{L}(\emptyset)$, is unitary and so the study of $\mathbb{E}(\emptyset)$ spaces simplifies the study of $\mathbb{L}(\emptyset)$ spaces. For further details of $\mathbb{E}(\emptyset)$ spaces, one can look into the papers of de Branges ([8], [9]).

1.4 Factorization Theorems

Though many factorization theorems for operator-valued analytic functions are available for serveying, but we mention here some references and results of the factorization theory which are analogous to some known results for H^p-functions ($1 \leq p \leq \infty$). In particular, we are interested in the factorization theory of bounded functions in the unit disc. A good deal of theory of H^p-functions is given in the book of Hoffman [15]. For operator-valued functions theory, one can look into the papers of Lax[17], Sarason [21], de Brangs and Rovnyak ([6], [7]), Sz.-Nagy and Foias [24] . By factorizations of a bounded (by 1) analytic function $B(z)$ in the unit disc, we mean to factor $B(z)$ into $A(z)$ and $C(z)$ such that $H(A)$ and $H(C)$ are non-null spaces. To indicate the nature of the factorization theory which is of interest to us, we quote a result of de Branges ([6]; p. 355), which is also used in Chapter 4.

Let $F(z)$ be an analytic function in the disc of radius $a > 1$ about the origin such that $1-F(z)$ assumes completely continuous values only, and $F(w)$ has dense range for some w, $|w| < 1$. Then there exist projections P_i, $i=1,2,\ldots,n$ of finite dimensional ranges such that

(1.4.1) $$F(z) = (1-P_0+P_1 z)\ldots(1-P_r+P_r z)\ G(z)$$

for some analytic function $G(z)$ such that $G(0)$ has an operator inverse.

It can be seen that the points, where $F(z)$ (defined by (1.4.1)) fails to have operator inverses, are isolated, but in general, it is not true if no completely continuous hypothesis is made. An example of such function can be constructed by applying the result of the Theorem 13 ([13] ; p. 74) (for a contraction operator which does not have discrete spectrum). So the study of operator-valued analytic functions is essentially different from that of complex analytic function theory. $H(B)$ spaces theory can be used as a tool in factorizations of operator-valued functions.

1.5 Invariant Subspaces

"Does every bounded linear operator defined in a Hilbert space posses a non-trivial invariant subspace " is among the fundamental problems of operator theory which are yet to be answered. There are infact many ways to strike upon this problem but the method which is of interest to us, is to find first its uniterily equivalent transformation (canonical model) in some known Hilbert space and then to test the existence of the invariant subspaces of equivalent transformation in the known space. Such canonical models of the given transformation are available in the papers of Beurling [2] , Rota [18] , Sz.-Nagy and Foias [25] . They also show that the problem reduces to factorizations of bounded operator-valued analytic functions. In fact, the problem can be reduced to the factorization of $B(z)$, when the multiplications by $B(z)$ and $B^*(z)$ are isometric transformations in $S(z)$. It equivalent to determine closed subspaces of $H(B)$ which contain $\frac{f(z)-f(0)}{z}$

alongwith $f(z)$. In general, it is not yet known but if some restrictions are imposed on coefficients of $B(z)$(e.g. $1-B(0)\overline{B}(0)$ is completely continuous)it is possible to have non-trivial factors. For detailed study of factorization theorems and invariant subspace we can refer to the books [24] and [13].

Work presented in the thesis

The present thesis can be divided broadly in three parts, (a) characterizations of certain Hilbert spaces of power series, (b) factorization theorems for operator-valued analytic functions and (c) applications of factorization theorems in proving the existence of invariant subspaces of linear transformations on Hilbert spaces under certain restrictions. We give chapterwise, a brief skech of the work presented in the thesis.

In Chapter II, H^2-space of complex-analytic functions is chracterized. An alternative proof of a characterization of $S(z)$ is given and $S(z)$ is studied in the light of reducing subspaces of the transformation $f(z) \rightarrow \frac{f(z)-f(0)}{z}$.

Chapter III mainly consists of some characterizations of $H(B)$ spaces when the equality is not attained in (1.2.4). Firstly, we assume S to be 1-dimensional and later on, any finite dimensional space.

We begin Chapter IV by extending some properties of $R(0)$ in $H(B)$, for the transformation defined by (1.2.10) in $\mathbb{D}(B)$. We give some factorization theorems for $B(z)$ when $H(B^*)$ does not contain any element of the form $B^*(z)c$ for any non-zero vector c. We also study

analytic continuation of $B(z)$ across the unit circle in the light of the spectrum of $R(0)$. The chapter ends with investigations of Blaschke products for operator-valued analytic functions in the upper half plane.

In the last chapter, we use factorization theorems for proving the existence of invariant subspaces of bounded operaters. The main result says that a contraction operator T defined in a Hilbert space has a non-trivial invariant subspace unless T^n and T^{*n} do not converges strongly to zero <u>together</u> as $n \to \infty$.

CHAPTER II

SOME CHARACTERIZATIONS OF SPACE $S(z)$

2.0 Summary of the chapter: This chapter deals with some characterizations of the space $S(z)$ which consists of formal power series whose coefficients are vectors belonging to a fixed Hilbert space S (which may be finite or infinite dimensional). In Theorem 1, the existence of zeros of functions contained in the space, is used to characterize the space. For this case the coefficient space is taken to be 1-dimensional. The extension of this result for vector-valued functions is given in Theorem 2. In Theorem 3, a different proof is given to generalize and extend the result of Theorem 14 ([5]; p. 38). In Lemma 1, a brief account of the spectrum of the transformation $f(z) \longrightarrow \dfrac{f(z) - f(0)}{z}$ defined in $S(z)$, is given. It is also shown that there is no non-null reducing subspace of the above transformation such that the spectrum of the restricted transformation, is a proper subset of the spectrum of the given transformation. This property has been used in Theorem 4 to construct a model of the above transformation defined in $S(z)$. The chapter is ended with a corollary which extends a result of A. Beurling [2].

2.1 Properties of $S(z)$: If the coefficients of power series in $S(z)$ are complex numbers then $S(z)$ turns out to be a wellknown space H^2 (Hardy space). Let U be the unit disc in the complex plane. The Hardy class H^2 is a Hilbert space of functions analytic in U such that

(2.1.1) $$\|f(z)\|_{H^2} = \lim_{r\to 1-} M_2(r;f) < \infty \quad ,$$

where $$M_2(r;f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta\right)^{\frac{1}{2}} \quad .$$

We give here some elementary informations ([20]; p. 332) about H^2-space, which are used in this chapter.

(a) An analytic function $f(z) = \Sigma a_n z^n$ (z in U) belongs to H^2, if and only if, $\Sigma |a_n|^2 < \infty$; in this case

(2.1.2) $$\|f(z)\|_{H^2} = (\Sigma|a_n|^2)^{\frac{1}{2}} \quad .$$

(b) If $f(z)$ belongs to H^2-space then $f^*(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta})$ exists almost everywhere on T (the boundary of U), f^* belongs to $L^2(T)$; the Fourier coefficients of f^* is a_n for $n \geq 0$ and is 0 for $n < 0$.

(c) The function $f(z)$ is Cauchy integral of f^*, i.e.

$$f(w) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f^*(z)}{z-w}\, dz \quad ,$$

where Γ is a positive oriented unit circle.

(d) Since $\frac{1}{1-\bar{w}z}$ belongs to H^2-space whenever $|w| < 1$, so

$$f(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f^*(e^{i\theta})}{1-we^{-i\theta}}\, d\theta$$

or,

(2.1.3) $$f(w) = \left\langle f(z), \frac{1}{1-\bar{w}z} \right\rangle_{H^2} \quad .$$

Therefore, $f(z) \longrightarrow f(w)$ is a continuous linear functional on H^2 for all complex number w, $|w| < 1$.

2.2 In this section we characterize a Hilbert space whose elements are power series with complex coefficients.

<u>THEOREM 1</u>. Let H be a non-null Hilbert space of power series which respresent complex-valued analytic functions in the unit disc and satisfy the following:

(I) For every α in U, multiplication by $\frac{1-\bar{\alpha} z}{z-\alpha}$ takes H isometrically on to H.

(II) For all α in U, $f(z) \rightarrow f(\alpha)$ is a continuous linear functional on H.

Then the space H is isometrically equal to $S(z)$.

It is an easy verification that (I) and (II) are also necessary in $S(z)$. The existence of (II) follows from (2.1.3) whereas (I) can be verified directly from the definition given in (2.1.1). It can be observed that the codimension of the domain of multiplication transformation is 1, and the adjoint of the multiplication transformation is an isometry on H.

<u>Proof:</u> Because of (II), for every complex number w in U there exists a unique element k(w,z) in H such that the identity

$$(2.2.1) \qquad f(w) = \langle f(z), k(w,z) \rangle$$

holds for all elements f(z) of H. We prove the theorem by showing that $k(w,z) = \frac{1}{1-\bar{w}z}$ for all w in U. If α is in U, the inequality

$$k(\alpha, \alpha) = \langle k(\alpha, z), k(\alpha, z) \rangle \geq 0$$

follows from non-negativity of the innerproduct. We assert that $k(\alpha, \alpha) > 0$ whenever $|\alpha| < 1$. Indeed, if $k(\alpha, \alpha) = \| k(\alpha, z)\|^2 = 0$ then $k(\alpha, z) \equiv 0$ which implies that

$$f(\alpha) = \langle f(z), k(\alpha, z) \rangle = 0$$

for every $f(z)$ belonging H. By (I), $\frac{1-\bar{\alpha} z}{z-\alpha} f(z)$ belongs to H whenever $f(z)$ belongs to H. Repeating the same arguement, $\left(\frac{1-\bar{\alpha} z}{z-\alpha}\right)^2 f(z)$ belong to H. By induction, $\left(\frac{1-\bar{\alpha} z}{z-\alpha}\right)^m f(z)$ belongs to H for all $m=0,1,2,\ldots$ As $f(z)$ is an analytic function, so it can vanish at α, arbitrary many times, if and only if, $f(z) \equiv 0$. But then H has no non-zero element which contradicts the hypothesis that H is non-null Hilbert space. Therefore $k(\alpha,\alpha)>0$ for all α in U.

Let S_α be the transformation of multiplication by $\frac{1-\bar{\alpha} z}{z-\alpha}$ defined on the closure of functions which vanish at α. Since S_α has a partially isometric extension on H and its range is whole of H, hence S_α^* (the adjoint of S_α) is an isometric transformation defined on H. Therefore

$$S_\alpha^* : g(z) \longrightarrow \frac{z-\alpha}{1-\bar{\alpha} z}\, g(z),$$

and

$$\left\langle \frac{1-\bar{\alpha} z}{z-\alpha} f(z),\ g(z) \right\rangle = \left\langle f(z),\ \frac{z-\alpha}{1-\bar{\alpha} z}\, g(z) \right\rangle$$

whenever $g(z)$ belongs to H and $f(z)$ is an element of H which vanishes at α. In particular, if $g(z) = k(w,z)$ for some w in U, then

$$\begin{aligned}\left\langle f(z),\ \frac{z-\alpha}{1-\bar{\alpha} z}\, k(w,z)\right\rangle &= \left\langle \frac{1-\bar{\alpha} z}{z-\alpha}\, f(z),\ k(w,z)\right\rangle \\ &= \frac{1-\bar{\alpha} w}{w-\alpha}\, f(w) \\ &= \frac{1-\overline{w\alpha}}{w-\bar{\alpha}} \left\langle f(z), k(w,z) - \frac{k(\alpha,z)k(w,\alpha)}{k(\alpha,\alpha)} \right\rangle.\end{aligned}$$

Therefore the identity

$$\left\langle f(z),\ \frac{z-\alpha}{1-\bar{\alpha} z}\, k(w,z) - \frac{1-\bar{w}\alpha}{\bar{w}-\bar{\alpha}}\left(k(w,z) - \frac{k(\alpha,z)k(w,\alpha)}{k(\alpha,\alpha)}\right)\right\rangle = 0$$

holds for all elements $f(z)$ of H which vanish at α. As the element

$$\frac{z-\alpha}{1-\bar{\alpha} z} k(w,z) - \frac{1-\alpha\bar{w}}{\bar{w}-\bar{\alpha}}\left(k(w,z) - \frac{k(\alpha,z)\,k(w,\alpha)}{k(\alpha,\alpha)}\right)$$

vanishes at α , so we can conclude that

$$\frac{z-\alpha}{1-\bar{\alpha} z} k(w,z) - \frac{1-\alpha\bar{w}}{\bar{w}-\bar{\alpha}}\left(k(w,z) - \frac{k(\alpha,z)k(w,\alpha)}{k(\alpha,\alpha)}\right) \equiv 0 .$$

The above identity can be written as

$$(2.2.2) \quad k(w,z)\,(1-\bar{w}z) = \frac{k(\alpha,z)\,(1-\bar{\alpha}z)\,k(w,\alpha)\,(1-\alpha\bar{w})}{(1-|\alpha|^2)\,k(\alpha,\alpha)} .$$

An immediate observation is that $k(w,z) = \frac{1}{1-\bar{w}z}$ satisfies the above identity for all w in U. We show that $k(w,z)$ is uniquely determined from (2.2.2). If $k_1(w,z)$ also satisfies the identity (2.2.2), then by (2.2.1),

$$k_1(\alpha,w) = \left\langle k_1(\alpha,z), \frac{1}{1-\bar{w}z}\right\rangle \text{and } \frac{1}{1-\bar{w}\alpha} = \left\langle \frac{1}{1-\bar{w}z}, k_1(\alpha,z)\right\rangle .$$

It implies that $k_1(\alpha,w) = \frac{1}{1-\bar{\alpha}w}$. Since w in U is arbitrary, it follows that $k_1(w,z) = \frac{1}{1-\bar{w}z}$. The theorem now follows by observing that the finite linear combinations of $k(w,z)$ for different w in U form dense sets in $S(z)$ and H both and have the same norms, which implies that H is isometrically equal to $S(z)$.

The result of Theorem 1 can be extended when S is an arbitrary dimensional space. By a zero of a vector-valued analytic function, we mean a point where it assumes the null vector. Such points are isolated ([14] ; Theorem 3.11.5).

THEOREM 2 : Let H be a non-null Hilbert space of vector-valued analytic functions in the unit disc. If,

(i) for every complex number α, $|\alpha|<1$, the transformation $f(z) \to f(\alpha)$ defined on H, is bounded and has dense range in S,

(ii) whenever $f(z)$ belongs to H and has a zero at α in the unit disc, the function $f(z)\,\frac{1-\bar{\alpha}z}{\alpha - z}$ is in the space and has the same norm as $f(z)$,

(iii) the function $\frac{z-\alpha}{1-\bar{\alpha}z}\,f(z)$ belongs to H whenever $f(z)$ belongs to H and $|\alpha| < 1$,

then H is equal isometrically to $S(z)$ with coefficient of its elements in S.

Proof: The theorem can be proved on the lines of the proof given for Theorem 1.

2.3 The result of this section generalizes and extends the result of Theorem 14 ([15]; p 38), where the coefficient space is 1-dimensional. Our approach is also different from that used for Theorem 14. Instead of showing the denseness of polynomials contained in the given space, we compute the reproducing kernels directly from the hypotheses. This result is also the converse of problem 68 ([5]; p. 34).

THEOREM 3: Let H be a non-null Hilbert space of formal power series with vector coefficients belonging to a fixed Hilbert space S. Let the transformation $f(z) \to \frac{f(z) - f(0)}{z}$ be defined in H and its adjoint be given by $f(z) \to zf(z)$. Then H is isometrically equal to $S(z)$ whose coefficient space S_0 is a closed subspace of S.

Proof: Let $T: f(z) \longrightarrow \frac{f(z) - f(0)}{z}$ be the given transformation and $\mathcal{S}_o$ be the range of $(I-T^*T)$ in H. Since $(I-T^*T)\, T^n f(z) = a_n$ where $f(z) = \Sigma\, a_n z^n$, $\mathcal{S}_o$ contains all coefficients of elements of H. We define a new norm on $\mathcal{S}_o$, which is given by $\|f(0)\| = |f(0)|$. This is a well-defined norm and does not depend on the choice of $f(z)$ and depends purely on the initial coefficient. Since $\mathcal{S}_o$ is a subspace of $\mathcal{S}$, there exists a partial isometry on $\mathcal{S}$ onto $\mathcal{S}_o$. Without loss of generality it is assumed that $\mathcal{S}_o$ is contained isometrically in $\mathcal{S}$. Therefore,

$$\begin{aligned} \langle f(z),\, c \rangle &= \langle T^*T\, f(z),\, c \rangle + \langle f(0),\, c \rangle \\ &= \bar{c}\, f(0) \end{aligned}$$

for all $f(z)$ of H and c of $\mathcal{S}_o$, and the identity

$$\left\| \frac{f(z) - f(0)}{z} \right\|^2 = \|f(z)\|^2 - |f(0)|^2 \tag{2.3.1}$$

holds for every elements $f(z)$ of H. Therefore, if $f(z) = \Sigma\, a_n z^n$ is in H, where a_n is in $\mathcal{S}_o$ for all n=0,1,2,... then

$$|a_n|^2 = \|T^n f(z)\|^2 - \|T^{n+1} f(z)\|^2 \quad .$$

Adding such relations for m=0,1,2,...,n; we get

$$0 \le \|T^{n+1} f(z)\|^2 = \|f(z)\|^2 - |a_o|^2 - |a_1|^2 - \cdots - |a_n|^2 \ .$$

Because of the arbitrariness of n, it follows that $\sum_0^\infty |a_n|^2 \le \|f(z)\|^2$.

So H is contained in some $\mathcal{S}(z)$ whose coefficient space is $\mathcal{S}_o$ and the inclusion of H into $\mathcal{S}(z)$ does not increase norms.

Since $J(0):f(z) \longrightarrow f(0)$ is a continuous (it follows from (2.3.1)) transformation from H into S_0 and T is bounded by 1, the series $f(w) = \sum_{n=0}^{\infty} w^n J(0)T^n f(z)$ is convergent whenever $|w| < 1$, and so $J(w):f(z) \longrightarrow f(w)$ is a continuous transformation on H into S_0 for every complex number w, $|w| < 1$. Its adjoint transformation is defined on S into H and is of form, $J(w)^*: c \longrightarrow k(w,z)c$ for some power series $k(w,z)$ with operator coefficients. If $f(z)$ is in H, then

$$(2.3.2) \qquad \bar{c}f(w) = \langle f(z), k(w,z)c \rangle$$

for all complex numbers w, $|w| < 1$ and c in S. We now compute the form of $k(w,z)$.

For each vector c when $|w| < 1$,

$$(2.3.3) \qquad \bar{c}\, \frac{f(w)-f(0)}{w} = \left\langle \frac{f(z)-f(0)}{z}, k(w,z)c \right\rangle$$

and

$$(2.3.4) \qquad \bar{c}\,(f(w) - f(0)) = \langle f(z), (k(w,z) - k(0,z))\, c \rangle$$

are equalities which hold for all $f(z)$ of H. Comparing them, we obtain the identity

$$\langle f(z), (k(w,z) - k(0,z) - \bar{w}z\, k(w,z))\, c \rangle = 0$$

which holds for all $f(z)$ of H. Since c and $f(z)$ are arbitrary, we have

$$k(w,z) - k(0,z) - \bar{w}z\, k(w,z) \equiv 0$$

or,

$$(2.3.5) \qquad k(w,z) = \frac{k(0,0)}{\ldots}$$

Again by the hypothesis of the Theorem,

$$\langle f(z), \frac{k(0,z) - k(0,0)}{z} c \rangle = \langle zf(z), k(0,z)c\rangle = 0$$

for all $f(z)$ of H and c of S. So $k(0,z) = k(0,0)$ is a constant function. Since

$$0 \leq \bar{c}\, k(0,0)c = \langle k(0,z)c, k(0,z)c\rangle = \langle k(0,0)c, k(0,0)c \rangle$$

for all c in S, hence $k(0,0)$ is a projection in S onto a closed subspace of S_0. Since for all $f(z)$ of H

$$\bar{c}\, f(0) = \langle f(z), k(0,z)c \rangle,$$

and S_0 is contained in H, hence the range of $k(0,0)$ is whole of S_0. Thus

$$\bar{c}\, f(w) = \langle f(z), \frac{c}{1-\bar{w}z} \rangle$$

whenever $f(z)$ is in H, c in S_0 and $|w| < 1$. Now the proof is completed if we use the arguments given for Theorem 1.

2.4 Now we study the reducing subspaces of the transformation $f(z) \longrightarrow \frac{f(z) - f(0)}{z}$ defined in $S(z)$.

<u>LEMMA 1</u>. There exists no nontrivial reducing subspace of the transformation $f(z) \longrightarrow \frac{f(z) - f(0)}{z}$ defined in $S(z)$ such that the spectrum of the restriction of the transformation on the reducing subspace, is a proper subset of the spectrum of the given transformation.

Proof. Firstly, we discuss the spectrum of the given transformation (say T) in $S(z)$. Since for a closed transformation T, λ belongs to the spectrum, if and only if, either the dimension of the kernel of $(T - \lambda I), > 0$ or, the codimension of the range of $(T - \lambda I), > 0$, hence the spectrum of T coincides with the open unit disc. The boundary of the unit disc forms the essential spectrum of T. (Essential spectrum is the set of points λ such that either the range of $(T- \lambda I)$ is closed, or if the range is closed then the dimension of the kernel and codimension of the range of $(T- \lambda I)$ are not finite). It is also seen that the closed span of eigenvectors is whole of $S(z)$.

Secondly, we characterize reducing subspaces of T. Let M be a reducing subspace of T. Then $M = M(B)$ for some power series $B(z)$ with operator coefficients such that the multiplication by $B(z)$ is a partial isometry in $S(z)$. Since $B(z)c$ belongs to M for every vector c in S, hence $\frac{B(z)-B(0)}{z}c$ belongs to $M(B)$. But by the definition of $S(z)$, $\frac{B(z)-B(0)}{z}c$ is orthogonal to $M(B)$, which means that $B(z) = B(0)$. Therefore M is a subspace of $S(z)$ whose elements have coefficients in the closure of the range of $1-B(0)\overline{B}(0)$. Infact, this proves that there exists one-one correspondence in the reducing subspaces of T and the closed subspaces of S. The Lemma now follows as it is easily seen that the restriction of $f(z) \longrightarrow \frac{f(z)-f(0)}{z}$ on M will have same spectrum as that of $f(z) \rightarrow \frac{f(z)-f(0)}{z}$ in $S(z)$.

Following corollary is a direct consequence of the Lemma.

COROLLARY: The transformation $f(z) \rightarrow \frac{f(z)-f(0)}{z}$ in $S(z)$ has no non-trivial reducing subspace if the coefficients of the elements of $S(z)$ are complex numbers.

2.4 Let T_k be a continuous linear transformation defined on a Hilbert space H_k into itself, k=1,2. The transformations T_1 and T_2 are called unitarily equivalent if there exists a unitary transformation U such that $UT_1 = T_2U$. As U preserves structural properties, it is possible to study a transformation T_1 in terms T_2 which can be described explicitly. An use of such a model was shown by A. Beurling [2], and its vector generalization was given by de Branges and Rovnyak [6] . We follow their approach in the following characterization of $S(z)$.

THEOREM 4. Let H be a Hilbert space and T be a bounded transformation on H into itself such that its adjoint is isometric. Assume that T has no non-trivial reducing subspace such that the spectrum of the restriction of T on the reducing subspace is a proper subset of the spectrum of T. Then H is isometrically equal to $S(z)$.

Proof: We can take for granted that there is no non-zero element f in H such that $\| T^n f \| = \| f \|$ for n=1,2,3,... Otherwise, if M is the closed span of such elements, then obviously M is invariant under T. As T^* is isometric, so M reduces T. The restriction of T on M is a unitary transformation. It is wellknown that a unitary transformation has reducing subspace for which the spectrum of the restricted transformation is an arbitrary closed proper subset of the spectrum of

the given transformation. But this contradicts the hypothesis, so M must be the null space.

Let S be a fixed Hilbert space whose dimension is not less than the dimension of the closure of the range of $(1-T^*T)$ in H. Therefore by Theorem 1 ([6] ; p. 347) T is unitarily equivalent to the transformation $f(z) \longrightarrow \frac{f(z)-f(0)}{z}$ in some Hilbert space H_0 of formal power series whose coefficients belong to S, such that the identity

$$(2.4.1) \qquad \left\| \frac{f(z)-f(0)}{z} \right\|_0^2 = \| f(z) \|_0^2 - |f(0)|^2 .$$

holds for all element $f(z)$ of H_0. We use T to denote the above transformation in H_0. By (2.4.1), the transformaton $f(z) \longrightarrow f(0)$ is continuous from H_0 into S. Further, as $(1-wT)^{-1}$ is a convergent power series in the operator algebra of S whenever $|w| < 1$, so $f(z) \longrightarrow f(w)$ is also continuous transformation into S for each complex number w, $|w| < 1$. Thus H_0 consists of power series which converge for each w, $|w| < 1$. It follows from the section 1.3 that $H_0 = \mathbb{L}(\emptyset)$ for some power series $\emptyset(z)$ whose coefficients opretaors such that Re $\emptyset(w) \geq 0$ when $|w| < 1$.

We suppose that polynomials contained in $\mathbb{L}(\emptyset)$ form a dense set in $\mathbb{L}(\emptyset)$. Infact , if $\mathbb{L}$ is the orthogonal complement of polynomials belonging to $\mathbb{L}(\emptyset)$ then $\mathbb{L}$ is a closed subspace of $\mathbb{L}(\emptyset)$ which reduces T because its orthogonal complement is invariant under T^* as well as T, both. Let $g(z)$ be an element orthogonal to the range of T^* contained in $\mathbb{L}$. Then,

$$0 = \langle g(z), T^*f(z) \rangle_{\mathbb{L}} = \langle Tg(z), f(z) \rangle_{\mathbb{L}}$$

for all $f(z)$ of $\mathbb{L}$, implies that $Tg(z)=0$, i.e., $g(z)=g(0)$. But as there is no non-zero constant vector in $\mathbb{L}$, $g(z)=g(0)=0$. This shows that the range of restriction of T on $\mathbb{L}$ is whole of $\mathbb{L}$, which means that restriction of T is unitary in $\mathbb{L}$. The arguement used in the beginning of the proof shows that $\mathbb{L}$ is the null space. Thus we arrive to conclude that polynomials contained in $\mathbb{L}(\emptyset)$ are dense in $\mathbb{L}(\emptyset)$.

If (2.4.1) is applied to all polynomials of $\mathbb{L}(\emptyset)$ we get that the polynomials are contained isometrically in $S(z)$ whose coefficient space is S. Thus $\mathbb{L}(\emptyset)$ is a closed subspace of $S(z)$ and the coefficients of its elements belong to the isometric isomorphic image of the range of $(1-T^*T)$ in S. As $\mathbb{L}(\emptyset)$ contains $\frac{f(z)-f(0)}{z}$ and $zf(z)$ both, whenever it contains $f(z)$ so by Theorem 3 of this chapter we conclude that $\mathbb{L}(\emptyset)$ is equal to some $S(z)$ and its coefficient space has dimension greater than or equal to 1. This completes the proof of the Theorem.

<u>COROLLARY</u>: In Theorem 4, T has a non-trivial reducing subspace, if and only if, the coefficient space of $S(z)$ has dimension greater than 1. The corollary follows from Lemma 1 and the proof of Theorem 4.

Application here shows that a Theorem of A. Beaurling [2], can be obtained as a corollary of our Theorem 4.

<u>Theorem</u> (Beurling). If T is a linear transformation on a Hilbert space H into itself such that

(A) the eigenvectors form a fundamental set in H,

(B) $||T|| \leq 1$, $||T^n f|| \to 0$ as $n \to \infty$ for all f of H,

(C) $||T^* f|| = ||f||$,

(D) atleast one eigenvalue is simple

then H is isometrically equal to $S(z)$ with complex coefficients.

CHAPTER III

CHARACTERIZATIONS OF H(B) SPACES

Summary of the chapter: As the title suggests, the chapter mainly consists of characterizations of H(B) spaces. In the first section, the case when the coefficients are complex numbers, is taken and H(B) is characterized when the power series B(z) belongs to it. In the second section, the result concerns with characterizations of H(B) spaces when the coefficient space $\mathcal{S}$ is finite dimensional and H(B) contains a non-zero element of the form B(z)c for some vector c.

3.1 The following known result ([6]; p.350) is given an alternate proof.

LEMMA 1: Let B(z) be a non-zero power series with operator coefficients. A necessary and sufficient condition that the identity

$$(3.1.1) \qquad \left\| \frac{f(z)-f(0)}{z} \right\|_B^2 = \| f(z) \|_B^2 - | f(0) |^2$$

holds for every element f(z) of H(B), is that H(B) does not contain any non-zero element of the form B(z)c for any non-zero constant c.

Proof: This is obtained as a corollary to Theorem 13 ([6]; p.354) which states that the transformation $f(z) \longrightarrow \hat{f}(z)$ defined on $H(B^*)$ into H(B) is an isometry, if and only if, there is no element of the form B(z)c in H(B) for any non-zero constant c, where

$$\bar{c}\,\hat{f}(w) = \langle f(z), \frac{B^*(z)-\bar{B}(w)}{z-\bar{w}}\, c \rangle_{B^*}$$

Beside this it also states that if $f(z) \longrightarrow \hat{f}(z)$ then $\left(zf(z) - B^*(z)\hat{f}(0)\right) \longrightarrow \frac{\hat{f}(z)-\hat{f}(0)}{z}$ under this transformation. Let $\mathbb{D}(B)$ be the extension space of $H(B)$. Since $f(z) \longrightarrow (f(z), \hat{f}(z))$ is an isometry into $\mathbb{D}(B^*)$, and for every $f(z)$ of $H(B)$ there exists a $g(z)$ in $H(B^*)$ such that $f(z)=\hat{g}(z)$, we have

$$\| f(z)\|^2_B = \| g(z)\|^2_{B^*} = \| (g(z), f(z))\|^2_{\mathbb{D}(B^*)}$$

$$= \| (zg(z)-B^*(z)f(0), \frac{f(z)-f(0)}{z})\|^2_{\mathbb{D}(B^*)} + |f(0)|^2$$

$$= \| \frac{f(z)-f(0)}{z}\|^2_B + |f(0)|^2$$

for all elements $f(z)$ of $H(B)$, if and only if, there is no non-zero element of the form $B(z)c$ for any vector c. Hence the Lemma.

(3.1.2) In this section, the coefficients of the elements of $H(B)$ spaces are complex numbers. A characterization of $H(B)$ spaces which do not contain $B(z)$ as a non-zero element was given by de Branges and Rovnyak ([5] ; p. 39). We obtain here similar results when $H(B)$ possesses $B(z)$ as a non-zero element.

<u>THEOREM 1</u>: Let $H(B)$ be a given space such that $B(z)$ belongs to $H(B)$. Then $\frac{f(z)-f(0)}{z}$ and $zf(z)$ belong to $H(B)$ whenever $f(z)$ belongs to $H(B)$, and the identity

$$(3.1.3)\qquad \|f(z)\|_B^2 - \left\|\frac{f(z)-f(0)}{z}\right\|_B^2 = |f(0)|^2 + |\langle f(z), e(z)\rangle_B|^2$$

holds for all elements $f(z)$ of $H(B)$, where $e(z)$ is some fixed element of $H(B)$.

Proof: For simplicity, we drop the suffix B. By the definition of $H(B)$ spaces ,

$$T: f(z) \longrightarrow \frac{f(z)-f(0)}{z}$$

is a bounded linear transformation in $H(B)$ and its adjoint

$$T^*: f(z) \longrightarrow zf(z) - B(z)\langle f(z), \frac{B(z)-B(0)}{z}\rangle .$$

Since $B(z)$ belongs to $H(B)$, we note that $zf(z)$ belongs to $H(B)$ for each $f(z)$ of $H(B)$. We also note that

$$(3.1.4)\qquad (1-T^*T): f(z) \longrightarrow f(0) + B(z)\langle Tf(z), TB(z)\rangle ,$$

and

$$\begin{aligned}\langle Tf(z), TB(z)\rangle &= \langle T^*Tf(z), B(z)\rangle \\ &= \langle f(z), B(z)\rangle - f(0)\langle 1, B(z)\rangle \\ &\quad -\langle Tf(z), TB(z)\rangle \|B(z)\|^2 ,\end{aligned}$$

or,

$$\langle Tf(z), TB(z)\rangle (1+\|B(z)\|^2) = \langle f(z), B(z)\rangle - f(0) \\ (\langle 1-B(z)\overline{B}(0), B(z)\rangle + \overline{B}(0)\,\|B(z)\|^2)$$

or,

$$\langle Tf(z),\ TB(z)\rangle = \frac{\langle f(z),B(z)\rangle}{(1+\|B(z)\|^2)} - \overline{B}(0)f(0)$$

on using (1.2.9). Substituting in (3.1.4), we get,

$$(1-T^*T):\ f(z) \longrightarrow (1-B(z)\overline{B}(0))f(0)+B(z)\ \frac{\langle f(z),B(z)\rangle}{(1+\|B(z)\|^2)}\ .$$

Therefore,

$$\|f(z)\|^2-\|Tf(z)\|^2 = f(0)\langle(1-B(z)\overline{B}(0)),f(z)\rangle+\frac{|\langle f(z),B(z)\rangle|^2}{(1+\|B(z)\|^2)}$$

$$= |f(0)|^2+\frac{|\langle f(z),B(z)\rangle|^2}{(1+\|B(z)\|^2)}\ .$$

Theorem 1 follows with $e(z) = \dfrac{B(z)}{(1+\|B(z)\|^2)^{\frac{1}{2}}}$.

We show that the hypothesis of Theorem 1 is also sufficient.

<u>THEOREM 2</u>: Let $\mathbb{H}$ be a Hilbert space of power series such that $\frac{f(z)-f(0)}{z}$ and $zf(z)$ belong to $\mathbb{H}$ whenever $f(z)$ is in $\mathbb{H}$. Assume that the identity

$$(3.1.5)\qquad \|f(z)\|^2-\left\|\frac{f(z)-f(0)}{z}\right\|^2 = |f(0)|^2+|\langle f(z),e(z)\rangle|^2\ ,$$

holds for all elements of $\mathbb{H}$, where $e(z)$ is some fixed non-zero element of $\mathbb{H}$ with $\|e(z)\|<1$. Then $\mathbb{H}$ is isometrically equal to some $H(B)$ such that $B(z)$ is a non-zero element of $H(B)$.

Proof: Let T and M denote the linear transformation $f(z) \longrightarrow \frac{f(z)-f(0)}{z}$ and $f(z) \longrightarrow zf(z)$ in $\mathbb{H}$, respectively. The identity (3.1.5) implies that T and M are bounded by 1 and $\frac{1}{(1-\|e(z)\|^2)^{\frac{1}{2}}}$ respectively.

Since $MT\,f(z) = f(z)-f(0)$, it follows that $f(0)$ belongs to $\mathbb{H}$ and in particular, 1 belongs to $\mathbb{H}$. If we write $B(z)=\frac{e(z)}{(1-\|e(z)\|^2)^{\frac{1}{2}}}$ for some element $B(z)$ of $\mathbb{H}$, the identity (3.1.5) can be written as

$$(3.1.6) \qquad \langle f(z),g(z)\rangle - \langle Tf(z), Tg(z)\rangle = f(0)\overline{g}(0) + \frac{\langle f(z),B(z)\rangle \langle B(z),g(z)\rangle}{(1+\|B(z)\|^2)}$$

for all $f(z)$ and $g(z)$ in $\mathbb{H}$. Therefore,

$$(3.1.7) \qquad \langle f(z), 1\rangle = f(0) + \frac{\langle f(z),B(z)\rangle \langle B(z),1\rangle}{(1+\|B(z)\|^2)} .$$

Putting $f(z) = B(z)$ in (3.1.7), we have,

$$B(0) = \frac{\langle B(z), 1\rangle}{(1+\|B(z)\|^2)} .$$

Therefore, from (3.1.7), we get

$$f(0) = \langle f(z), 1-B(z)\overline{B}(0)\rangle$$

for all $f(z)$ in $\mathbb{H}$, which implies that $J: f(z) \longrightarrow f(0)$ is a continuous linear functional on $\mathbb{H}$. Since T is bounded by 1, $\sum_0^\infty w^n T^n f(z)$ converges in the metric of $\mathbb{H}$ whenever $f(z)$ is in $\mathbb{H}$ and $|w| < 1$. Because J is a contraction, the series $f(w) = \sum_0^\infty w^n J(T^n f(z))$ converges

when $|w| < 1$. This implies that $J(w): f(z) \longrightarrow f(w)$ is a continuous linear functional on $\mathbb{H}$ for all complex number w, $|w| < 1$. Therefore there exists $k(w,z)$ in $\mathbb{H}$ such that

$$f(w) = \langle f(z), k(w,z) \rangle \tag{3.1.8}$$

for all elements $f(z)$ of $\mathbb{H}$, when $|w| < 1$.

Using (3.1.6), we have,

$$\begin{aligned} \langle f(z), TB(z) \rangle &= \langle TMf(z), TB(z) \rangle \\ &= \frac{\langle Mf(z), B(z) \rangle}{(1 + \|B(z)\|^2)} \end{aligned} \tag{3.1.9}$$

for all $f(z)$ in $\mathbb{H}$. We now compute T^* with the aid of (3.1.6) and (3.1.9). Thus

$$\begin{aligned} \langle T^*f(z), g(z) \rangle &= \langle TMf(z), Tg(z) \rangle \\ &= \langle Mf(z), g(z) \rangle - \frac{\langle Mf(z), B(z) \rangle \langle B(z), g(z) \rangle}{(1 + \|B(z)\|^2)} \\ &= \langle Mf(z), g(z) \rangle - \langle \langle f(z), TB(z) \rangle B(z), g(z) \rangle . \end{aligned}$$

Therefore, $T^*: f(z) \longrightarrow zf(z) - \langle f(z), TB(z) \rangle B(z)$.

Since, $\langle f(z), k(w,z) \rangle = f(w)$

$$\begin{aligned} &= f(0) + w \frac{f(w) - f(0)}{w} \\ &= \langle f(z), 1 - B(z)\overline{B}(0) \rangle + \langle Tf(z), \overline{w}\, k(w,z) \rangle \\ &= \langle f(z), 1 - B(z)\overline{B}(0) + \overline{w}\, T^*k(w,z) \rangle , \end{aligned}$$

for all elements $f(z)$ of $\mathbb{H}$, we get

$$k(w,z) = (1-B(z)\overline{B}(0) + \overline{w}\ T^{*}k(w,z))$$

$$= (1-B(z)\overline{B}(0)) + \overline{w}zk(w,z) - \overline{w}\langle k(w,z),\ TB(z)\rangle B(z),$$

or, $$(1-z\overline{w})\ k(w,z) = (1-B(z)\overline{B}(0)) - \overline{w}\ \frac{\overline{B}(w)-\overline{B}(0)}{\overline{w}}\ B(z)$$

or, $$k(w,z) = \frac{1-B(z)\overline{B}(w)}{(1-z\overline{w})} \quad .$$

Thus, we have shown that if $|w| < 1$, $\dfrac{1-B(z)\overline{B}(w)}{(1-z\overline{w})}$ belongs to $\mathbb{H}$ and that

$$f(w) = \left\langle f(z),\ \frac{1-B(z)\overline{B}(w)}{1-z\overline{w}} \right\rangle$$

holds for all $f(z)$ in $\mathbb{H}$. Taking $f(z) = \dfrac{1-B(z)\overline{B}(w)}{(1-\overline{w}z)}$, we get,

$$0 \leq \left\| \frac{1-B\ z)\overline{B}(w)}{(1-\overline{w}z)} \right\|^{2} = \frac{1-|B(w)|^{2}}{1-|w|} \quad .$$

It follows that, $|B(w)| \leq 1$ whenever $|w| < 1$. Therefore a space $H(B)$ exists. By (1.2.9), $\dfrac{1-B(z)\overline{B}(w)}{(1-z\overline{w})}$ belongs to $H(B)$ for all complex number w, $|w| < 1$, and

$$f(w) = \left\langle f(z),\ \frac{1-B(z)\overline{B}(w)}{(1-\overline{w}z)} \right\rangle_{B}$$

for all elements $f(z)$ of $H(B)$. The Theorem now follows as the finite linear combinations of elements of the form $k(w,z)$ form dense sets and have equal norms in $\mathbb{H}$ and $H(B)$.

We now generalise the identity (3.1.3) and obtain analogues of problems 87, 88 and 89 ([5] ; p.46).

<u>COROLLORY 1</u>: Let H(B) be a given space such that B(z) belongs to H(B). Then

$$f(\alpha)\bar{g}(\beta) + \frac{1}{(1+\|B(z)\|_B^2)}\left\langle \frac{f(z)-f(\alpha)}{z-\alpha}, B(z)\right\rangle_B \left\langle B(z), \frac{g(z)-g(\beta)}{z-\beta}\right\rangle_B$$

$$= (f(z)\,0\,g(z)) + \alpha\left(\frac{f(z)-f(\alpha)}{z-\alpha}\,0\,g(z)\right)$$

$$+ \bar{\beta}\left(f(z)\,0\,\frac{g(z)-g(\beta)}{z-\beta}\right)$$

$$- (1-\alpha\bar{\beta})\left(\frac{f(z)-f(\alpha)}{z-\alpha}\,0\,\frac{g(z)-g(\beta)}{z-\beta}\right)$$

for all f(z) and g(z) in H(B) when $|\alpha| < 1$ and $|\beta| < 1$, where

$$(f(z)\,0\,g(z)) = \langle f(z), g(z)\rangle_B - \frac{1}{(1+\|B(z)\|_B^2)}\langle f(z), B(z)\rangle_B \langle B(z), g(z)\rangle_B.$$

<u>Proof</u>: We drop the suffix B. If we note that $R(\alpha) = R(0)(1-\alpha R(0))^{-1}$ for all α, $|\alpha| < 1$, the identity can be reduced to the following form.

$$f(\alpha)\bar{g}(\beta) = \left\langle \sum_0^\infty \alpha^n R(0)^n f(z), g(z)\right\rangle$$

$$- \left\langle R(0)\sum_0^\infty \alpha^n R(0)^n f(z),\ R(0)\sum_0^\infty \beta^m R(0)^m g(z)\right\rangle$$

$$+ \left\langle \sum_0^\infty \alpha^n R(0)^n f(z),\ \sum_1^\infty \beta^m R(0)^m g(z)\right\rangle$$

$$- \frac{1}{(1+\|B(z)\|^2)}\langle f(z), B(z)\rangle \left\langle B(z), \sum_0^\infty \beta^m R(0)^m g(z)\right\rangle$$

$$+ \left\langle \sum^\infty \alpha^n R(0)^n f(z), B(z)\right\rangle \left\langle B(z), \sum^\infty \beta^m R(0)^m g(z)\right\rangle.$$

We now generalise the identity (3.1.3) and obtain analogues of problems 87, 88 and 89 ([5] ; p.46).

COROLLORY 1: Let H(B) be a given space such that B(z) belongs to H(B). Then

$$f(\alpha)\,\overline{g}(\beta) + \frac{1}{(1+\|B(z)\|_B^2)} \left< \frac{f(z)-f(\alpha)}{z-\alpha}, B(z) \right>_B \left< B(z), \frac{g(z)-g(\beta)}{z-\beta} \right>_B$$

$$= (f(z) \circ g(z)) + \alpha \left(\frac{f(z)-f(\alpha)}{z-\alpha} \circ g(z) \right)$$

$$+ \overline{\beta} \left(f(z) \circ \frac{g(z)-g(\beta)}{z-\beta} \right)$$

$$- (1-\alpha\overline{\beta}) \left(\frac{f(z)-f(\alpha)}{z-\alpha} \circ \frac{g(z)-g(\beta)}{z-\beta} \right)$$

for all f(z) and g(z) in H(B) when $|\alpha| < 1$ and $|\beta| < 1$, where

$$(f(z) \circ g(z)) = \left< f(z), g(z) \right>_B - \frac{1}{(1+\|B(z)\|_B^2)} \left< f(z), B(z) \right>_B \left< B(z), g(z) \right>_B,$$

Proof: We drop the suffix B. If we note that $R(\alpha) = R(0)(1-\alpha R(0))^{-1}$ for all α, $|\alpha| < 1$, the identity can be reduced to the following form.

$$f(\alpha)\overline{g}(\beta) = \left< \sum_0^\infty \alpha^n R(0)^n f(z), g(z) \right>$$

$$- \left< R(0) \sum_0^\infty \alpha^n R(0)^n f(z),\ R(0) \sum_0^\infty \beta^m R(0)^m g(z) \right>$$

$$+ \left< \sum_0^\infty \alpha^n R(0)^n f(z),\ \sum_1^\infty \beta^m R(0)^m g(z) \right>$$

$$- \frac{1}{(1+\|B(z)\|^2)} \left< f(z), B(z) \right> \left< B(z), \sum_0^\infty \beta^m R(0)^m g(z) \right>$$

$$+ \left< \sum \alpha^n R(0)^n f(z), B(z) \right> \left< B(z), \sum^\infty \beta^m R(0)^m g(z) \right> .$$

If $f(z) = \Sigma a_n z^n$, $g(z) = \Sigma b_m z^m$ then by equating the coefficients of $\alpha^n \bar{\beta}^m$, we get

$$a_n \bar{b}_m = - \langle R(0)^{n+1} f(z), R(0)^{m+1} g(z) \rangle + \langle R(0)^n f(z), R(0)^m g(z) \rangle$$

$$- \frac{1}{(1+ \| B(z) \|^2)} \langle R(0)^n f(z), B(z) \rangle \langle B(z), R(0)^m g(z) \rangle$$

which follows from (3.1.3) for all $f(z)$ and $g(z)$ in $H(B)$ and integers $n \geq 0$, $m \geq 0$.

<u>COROLLORY 2</u>: Let $H(B)$ be a given space containing $B(z)$. Then

$$\left\langle \frac{B(z)-B(\alpha)}{z-\alpha}, \frac{B(z)-B(\beta)}{z-\beta} \right\rangle_B = \frac{1-B(\alpha)\overline{B}(\beta)}{1-\alpha\bar{\beta}}$$

$$- \frac{1}{(1+\| B \|^2)(1-\alpha\bar{\beta})} \left(\left\langle \frac{B(z)-B(\alpha)}{z-\alpha}, B(z) \right\rangle_B + 1 \right)$$

$$\left(\left\langle B(z), \beta \frac{B(z)-B(\beta)}{z-\beta} \right\rangle_B + 1 \right).$$

<u>Proof</u>: Putting $f(z) = g(z) = (1-B(z)\overline{B}(0))$ in Corollory 1, and simplifying, we get the result of corollory 2.

3.2 A characterization of H(B) spaces over an infinite dimensional coefficient space $\mathcal{S}$, is given in Theorem 11 ([9]; p. 171.). In this section, we obtain a similar characterization for the remaining case, i.e. when $\mathcal{S}$ is finite dimensional. In Theorem 3, the necessity of the hypothesis is shown whereas the sufficiency is obtained in Theorem 4. As the same proofs work in the case of infinite dimensional coefficient space, there is no bar on the coefficient space.

THEOREM 3: In a given space H(B), the sum of the dimension of the closure of the range of $(1-R(0)R(0)^*)$ and the dimension of the closed span of constants c such that

$$(3.2.0) \qquad \sup \left(|c+f(0)|^2 + \left\| \frac{f(z)-f(0)}{z} \right\|_B^2 - \| f(z) \|_B^2 \right) < \infty ,$$

where the supremum is taken over all elements f(z) of H(B), does not exceed the dimension of the coefficient space.

Proof: If the coefficient space is infinite dimensional, the Theorem follows trivially as the dimension of H(B) space does not exceed the dimension of $\mathcal{S}(z)$ which has the same dimension as that of the coefficient space. So we assume that the given coefficient space is finite dimensional. It follows from (1.2.5) that

$$(1-R(0)R(0)^*): f(z) \longrightarrow \frac{B(z)-B(0}{z} \tilde{f}(0)$$

for every element f(z) of H(B), where $\tilde{f}(0)$ is a vector defined in (1.2.6). Hence the dimension of the closure of the range of $(1-R(0)R(0)^*)$ does not exceed the dimension of the closed span of vectors of the form $\tilde{f}(0)$.

It is verified from the hint given in Problem 81 ([5] ; p.45) that a constant c satisfies (3.2.0), if and only if, $c=B(0)a$ for some constant a such that $\frac{B(z)-B(0)}{z}a = 0$. In that case, $\bar{a}\tilde{f}(0)=0$ for every element $f(z)$ of $H(B)$. Hence the dimension of the closed span of constants which satisfy (3.2.0), is not greater than the dimension of the orthogonal complement of the constants of the form $\tilde{f}(0)$ obtained by (1.2.6) for every element $f(z)$ of $H(B)$. This completes the proof.

<u>THEOREM 4</u>: Let H_0 be a Hilbert space of formal power series with vector coefficients such that the transformation $R(0):f(z) \to \frac{f(z)-f(0)}{z}$ is defined in H_0, and

$$\left\| \frac{f(z)-f(0)}{z}\right\|_0^2 \leq \| f(z)\|_0^2 - |f(0)|^2$$

whenever $f(z)$ belongs to H_0. Assume that the sum of the dimension of closure the range of $(1-R(0)R(0)^*)$ and the dimension of the closed span of constants c such that

$$\text{(3.2.1)} \qquad \sup \ |c+f(0)|^2 + \left\|\frac{f(z)-f(0)}{z}\right\|_0^2 - \|f(z)\|_0^2 < \infty,$$

where the supremum is taken over all elements $f(z)$ of H_0, does not exceed the dimension of the coefficient space. Then H_0 is equal isometrically to some $H(B)$ space.

<u>Proof</u>: The proof generalizes the proof of Theorem 6([6] ; p. 351) and extends the result of Theorem 11 ([9] ; p.171). Let $f(z) = \Sigma a_n z^n$ be in H_0. If we define $f_0(z)=f(z)$,

$$f_{n+1}(z) = \frac{f_n(z)-f_n(0)}{z} \quad \text{for } n=0,1,2\ldots, \text{ then}$$

$$\|f_{n+1}(z)\|_0^2 \leq \|f(z)\|_0^2 - |a_0|^2 - |a_1|^2 - \ldots - |a_n|^2 ,$$

for all $n \geq 0$. By the arbitrariness of n, it follows that $f(z)$ belongs to $\mathcal{S}(z)$ and $\|f(z)\| \leq \|f(z)\|_0$. Therefore H_0 is contained in $\mathcal{S}(z)$ and the inclusion of H_0 in $\mathcal{S}(z)$ does not increase norms. Since $f(z) \longrightarrow f(w)$ is a continuous transformation on $\mathcal{S}(z)$ into $\mathcal{S}$ for all complex number w, $|w| < 1$, hence $f(z) \longrightarrow f(w)$ is also a continuous transformation defined on H_0 into $\mathcal{S}$, whenever $|w| < 1$. The adjoint of the above transformation takes $c \longrightarrow k_0(w,z)c$ into H_0, where $k_0(w,z)$ is a power series with operator coefficients such that

$$(3.2.2) \qquad \bar{c}\, f(w) = \langle f(z), k_0(w,z)c \rangle_0$$

for all elements $f(z)$ of H_0 and c in $\mathcal{S}$, when $|w| < 1$. Let H_1 be the set of of power series $f(z)$ with vector coefficients such that $\frac{f(z)-f(0)}{z}$ belongs to H_0. Then H_1 is a Hilbert space in the 1-norm defined by the identity

$$(3.2.3) \qquad \|f(z)\|_1^2 = \left\| \frac{f(z)-f(0)}{z} \right\|_0^2 + |f(0)|^2.$$

If $f(z)$ is in H_1, define a new 2-norm by

$$\|f(z)\|_2^2 = \sup \left(\|f(z)+g(z)\|_1^2 - \|g(z)\|_0^2 \right) ,$$

where the supremum is taken over all elements $g(z)$ of H_0. The set of elements $f(z)$ of H_1 which have finite 2-norm, is a Hilbert space H_2 in 2-norm and is contained in H_1 such that its inclusion in H_1 does

not increase norms. Since H_1 is contained in $S(z)$ and the inclusion does not increase norm, it follows that the transformation $f(z) \longrightarrow f(w)$ on H_2 into S is continuous whenever $|w| < 1$ and its adjoint takes $c \longrightarrow k_2(w,z)c$, where $k_2(w,z)$ is a power series with operator coefficients, such that the identity

$$(3.2.4) \qquad \bar{c}\, f(w) = \langle f(z), k_2(w,z)c \rangle_2$$

holds for all elements $f(z)$ of H_2, when $|w| < 1$.

We show that the dimension of H_2 is not greater than the dimension of the coefficient space S. By the definition of 1-norm, the space S is contained isometrically in H_1 and its orthogonal complement in H_1 is the image of H_0 under the isometry $f(z) \longrightarrow zf(z)$. Let I_0 and I_2 be the inclusions of H_0 and H_2 in H_1, respectively. Then for each element $f(z)$ of H_0, $zf(z)$ belongs to H_1 and

$$\begin{aligned} \langle I_0^* zf(z), g(z) \rangle_0 &= \langle zf(z), g(z) \rangle_1 \\ &= \langle f(z), \frac{g(z)-g(0)}{z} \rangle_0 \\ &= \langle R(0)^* f(z), g(z) \rangle_0 \end{aligned}$$

whenever $g(z)$ is an element of H_0, which means that $I_0^* zf(z) = R(0)^* f(z)$. Let $zf(z) = u(z) + (zf(z)-u(z))$ be the minimal decomposition of $zf(z)$ as an element of H_1 with $u(z)$ in H_2 and $(zf(z)-u(z))$ in H_0 such that

$$(3.2.5) \qquad \| zf(z) \|_1^2 = \| u(z) \|_2^2 + \| zf(z)-u(z) \|_0^2 .$$

The elements $u(z)$ and $(zf(z)-u(z))$ are obtained from $zf(z)$ under the adjoints of I_2 and I_0, respectively. So (3.2.5) can be written in the

following form.

$$(3.2.6) \qquad \| u(z)\|_2^2 = \langle f(z), (1-R(0)R(0)^*)f(z)\rangle_0 \quad .$$

Let S_0 be the range of $(1-R(0)R(0)^*)$ in H_0 but with a new 3-norm. If $a(z) = (1-R(0)R(0)^*)f(z)$ and $b(z) = (1-R(0)R(0)^*)\, g(z)$ are in S_0, then define

$$\langle a(z), b(z)\rangle_3 = \langle f(z), (1-R(0)R(0)^*)g(z)\rangle_0 \quad .$$

It is easily checked that the above inner-product is well defined and does not depend on the choice of $f(z)$ and $g(z)$. By the dimension hypothesis, one can assume without loss of generality that S_0 is contained isometrically in the coefficient space S. Therefore it is possible to associate with each element $f(z)$ of H_0, a vector $\tilde{f}(0)$ depending on $f(z)$ such that $\| a(z)\| = |\tilde{f}(0)|$. Relating it with (3.2.6), we get, $\| u(z)\|_2 = |\tilde{f}(0)|$. If $\mathfrak{U}$ denotes the set of elements $u(z)$ obtained from $zf(z)$ under the adjoint I_2^* for each element $f(z)$ of H_0, then the dimension of the closure of $\mathfrak{U}$ in H_2, is not greater than the dimension of the closure of the range of $(1-R(0)R(0)^*)$ in H_0.

If $g(z)$ is in H_2 then by the minimal decomposition theory,

$$\langle zf(z), g(z)\rangle_1 = \langle u(z), g(z)\rangle_2 + \langle zf(z)-u(z), 0 \rangle_0$$

or,
$$\langle f(z), \frac{g(z)-g(0}{z} \rangle_0 = \langle u(z), g(z)\rangle_2$$

for every element $f(z)$ of H_0, where $u(z) = zf(z)-R(0)^*f(z)$. Therefore the orthogonal complement of $\mathfrak{U}$ in H_2 is the closed span of constants.

By the definition of 2-norm, $|c| \leq \|c\|_2$ whenever c belongs to H_2. It implies that the dimension of the closed span of constants contained in H_2, does not exceed the dimension of the closed span of constants satisfying (3.2.1). Therefore, it follows from the hypothesis that the dimension of H_2 is not greater than the dimension of the coefficient space. Consequently, there exists a partially isometric transformation on S onto H_2. It is clearly of the form $c \rightarrow B(z)c$ where $B(z)$ is a power series with operator coefficients. Since $B(z)c$ belongs to $S(z)$ for every constant c, and the inclusion of H_2 in $S(z)$ does not increase norms, so $B(z)c$ converges in the unit disc for every constant c in S. To compute $k_2(w,z)$, let $f(z)$ be in H_2, then $f(z) = B(z)a$ for some a in S and

$$\bar{c}\, f(w) = \bar{c}\, B(w)\, a = \langle B(z)a,\ B(z)\bar{B}(w)c \rangle_2$$

$$= \langle f(z),\ B(z)\bar{B}(w)c \rangle_2 \quad .$$

By the arbitrariness of c in S, $k_2(w,z) = B(z)\bar{B}(w)$. By the theory of minimal decomposition, the identity

$$\bar{c}\, f(w) = \langle f(z),\ k_o(w,z)c + k_2(w,z)c \rangle_1 \tag{3.2.7}$$

holds for each $f(z)$ in H_1 and c in S, whenever $|w| < 1$. Since $f(z) \rightarrow zf(z)$ is an isometry on H_o onto the orthogonal complement of S in H_1, the element $(c + z\bar{w}\, k_o(w,z)c)$ belongs to H_1 for each c in S when $|w| < 1$, and

$$\bar{c}\, f(w) = \langle f(z),\ c + z\bar{w}\ k_o(w,z)c \rangle_1 \tag{3.2.8}$$

for every element $f(z)$ of H_1. Comparing (3.2.8) with (3.2.7), we get,

$$k_0(w,z) + k_2(w,z) = 1 + z\bar{w}\, k_0(w,z),$$

or,

$$k_0(w,z) = \frac{1-B(z)\bar{B}(w)}{1-z\bar{w}}. \tag{3.2.9}$$

Since the inequality

$$\bar{c}\, k_0(w,w)c = \|k_0(w,z)c\|_0^2 \geq 0$$

holds for all c in S, it follows that $\|B(w)\| \leq 1$ whenever $|w| < 1$. This insures the existence of $H(B)$ whose elements are formal power series with vector coefficients. Since the finite sum of the elements of the form given by (3.2.9) form dense sets in H_0 and $H(B)$ and have same norms, it follows that H_0 is isometrically equal to $H(B)$. Hence the Theorem.

CHAPTER IV

FACTORIZATION THEOREMS

Summary of the Chapter: The chapter mainly consists of some factorization Theorems for an operator-valued function $B(z)$ such that the space $H(B)$ exists and

(4.0.1) $H(B^*)$ contains no non-zero element of the form $B^*(z)c$ for any non-zero vector c.

These results are analogous to those obtained by de Branges and Rovnyak when the multiplication by $B^*(z)$ in $S(z)$ is an isometry. It is well known that the isometry of the multiplication by $B^*(z)$ in $S(z)$ is merely sufficient to ensure the existence of (4.0.1), so it is worth obtaining the corresponding results under weaker condition, like (4.0.1). In the first theorem, a canonical model of a bounded transformation is obtained which suits the transformation defined by (1.2.10) in the extension space $\mathcal{D}(B)$ of $H(B)$, and is studied in Theorem 2 in case $H(B^*)$ is isometrically contained in $S(z)$. Later, we obtain a factorization of $B(z)$ when the equality is not attained in (1.2.4) atleast for one element of $H(B)$. Next, we relate the spectrum of the difference quotient transformation in $H(B)$ with the zeros (points where $B(z)$ fails to have bounded operator inverse) of $B(z)$ and determine further that $B(z)$ has analytic continuation to a point on the unit circle, if and only if, the point belongs to the resolvent set of $R(0)$. In Theorem 6, a sufficient

condition on the spectrum is obtained for the isolation of the zeros of $B(z)$. In the last section of the chapter, we obtain Blaschke product of operator-valued analytic functions in the upper half plane, following Nevanlinna's theory for complex analytic functions.

4.1 We determine necessary conditions of existence of the difference quotient transformation in $\mathbb{D}(B)$.

THEOREM 1: Let T be a transformation defined on a Hilbert space H into itself. Assume that T is bounded by 1 and there is no non-zero element x in H such that $\|T^n x\| = \|x\| = \|T^{*n} x\|$ for every $n=1,2,\ldots$ Then T is unitarily equivalent to the transformation

$$(4.1.1) \qquad (f(z),g(z)) \rightarrow (h(z),k(z))$$

defined on a Hilbert space $\mathbb{D}$ whose elements are pairs of power series $(f(z),g(z))$ with coefficients in a Hilbert space S such that

$$h(z) = \frac{f(z)-f(0)}{z},$$

$$\|(h(z),k(z))\|^2_{\mathbb{D}} = \|(f(z),g(z))\|^2_{\mathbb{D}} - |f(0)|^2$$

and if $(f(z),g(z)) \rightarrow (u(z),v(z))$ under the adjoint of the transformation defined by (4.1.1) then $v(z) = \frac{g(z)-g(0)}{z}$ and

$$\|(u(z),v(z))\|^2_{\mathbb{D}} = \|(f(z),g(z))\|^2_{\mathbb{D}} - |g(0)|^2 .$$

Proof: Let S_0 and S_1 be the ranges of $(1-T^*T)$ and $(1-TT^*)$, respectively, but with following different innerproducts: Define

$\langle a,b \rangle_0 = \langle f, (1-T^*T)g \rangle$ and $\langle c,d \rangle_1 = \langle f,(1-TT^*)_g \rangle$, where $a = (1-T^*T)f$, $b=(1-T^*T)g$, $c=(1-TT^*)f$ and $d=(1-TT^*)g$. As it is easily checked that the inner-products do not depend on the choice of the elements f and g in H, so S_0 and S_1 are well defined inner-products spaces with 0 and 1 inner-products, respectively. Let S be the completion of $(S_0 \times S_1)$in the norm

$$| (x,y)|^2 = \|x\|_0^2 + \|y\|_1^2 \; .$$

We will see that S is the required coefficient space. So we can assume without loss of generality that S_0 and S_1 are contained isometrically in S. If x is in H, define power series $f(z)= \Sigma a_n z^n$ and $g(z)= \Sigma b_n z^n$ with coefficients $a_n = (I-T^*T)T^n x$ and $b_n = (I-TT^*)\ T^{*n}x$ for each n=0,1,2.. Then

$$|a_n|^2 = \|T^n x\|^2 - \|T^{n+1}x\|^2 \quad \text{and} \quad |b_n|^2 = \|T^{*n}x\|^2 - \|T^{*n+1}x\|^2$$

for each n=0,1,2,... Since there is no non-zero element x in H such that $\|T^n x\| = \|x\| = \|T^{*n}x\|$ for all n=1,2..., hence f(z) and g(z) are zero simultaneously, if and only if, x is the null element. Let $\mathbb{D}$ be the image of the transformation $U:x \longrightarrow (f(z),g(z))$ defined over H. Then $\mathbb{D}$ is a Hilbert space in the unique inner product which makes U an isometry. By definition, if $U x \longrightarrow (f(z),g(z))$ and $U Tx \longrightarrow (h(z),k(z))$, then $h(z) = \dfrac{f(z)-f(0)}{z}$ and

$$\| (h(z),k(z)) \|_{\mathbb{D}}^2 = \|(f(z),g(z))\|_{\mathbb{D}}^2 - |f(0)|^2 \; .$$

Similarly, if $Ux \longrightarrow (f(z),g(z))$ and $U T^*x \longrightarrow (u(z),v(z))$ then $v(z) = \dfrac{g(z)-g(0)}{z}$. Since U is an unitary transformation,

$(f(z),g(z)) \longrightarrow (u(z),v(z))$ under the adjoint of the transformation defined by (4.1.1). The last norm identity follows by the definition of $\mathbb{D}$.

<u>COROLLARY</u>: In addition to the hypotheses of the Theorem , assume that $T^{*n} \longrightarrow 0$ as $n \to \infty$ in the norm. Then H is isometrically isomorphic to a closed subspace of $\mathcal{S}(z)$.

<u>Proof</u>: Recall from the proof of the theorem that if $g(z) = \Sigma\, b_n z^n$ then

$$|b_n|^2 = \|T^{*n}x\|^2 - \|T^{n+1}x\|^2 ,$$

for n=0,1,2.... Summing the identity for n=0,1,2...,N we get,

$$\| f(z),g(z) \|^2 = \|T^{*N}x\|^2 + |b_0|^2 + \ldots + |b_{N-1}|^2 .$$

Since $T^{*n}x \to 0$ as $n \to \infty$, hence $\|(f(z),g(z))\|_{\mathbb{D}} = \|g(z)\|^2$.

It implies that the set of elements $g(z)$ such that $(f(z),g(z))$ belongs to $\mathbb{D}$ is a closed subspace of $\mathcal{S}(z)$. Hence the corollary.

4.1.2 The following theorem extends the result of Theorem 12 ([6]; p. 357) for $\mathbb{D}(B)$ spaces.

<u>THEOREM 2</u>. Let H(B) be a given space with the transformation $R(0): f(z) \to \dfrac{f(z)-f(0)}{z}$ and the extension space $\mathbb{D}(B)$. If T denotes the transformation

$$(f(z),g(z)) \longrightarrow \left(\frac{f(z)-f(0)}{z} ,\; zg(z) - B^*(z)f(0) \right)$$

in $\mathbb{D}(B)$, then $R(0)^{*n} \to 0$ as $n \to \infty$ in H(B), if and only if,

$T^{*n} \longrightarrow 0$ as $n \to \infty$ in $\mathbb{D}(B)$.

Proof. (Sufficiency) Note that the transformation $f(z) \longrightarrow (f(z), \tilde{f}(z))$ is an isometry defined over $H(B)$ into $\mathbb{D}(B)$, where

$$\bar{c}\, \tilde{f}(\bar{w}) = \langle f(z), \frac{B(z)-B(w)}{z-w} c \rangle_B$$

for all c in S when $|w| < 1$. If $f(z)$ belongs to $H(B)$ and $\tilde{f}(z) = \Sigma b_n z^n$, then

$$\| T^{*n}(f(z), \tilde{f}(z)) \|^2_{\mathbb{D}(B)} = \| (f(z), \tilde{f}(z)) \|^2_{\mathbb{D}(B)} - |b_0|^2 - \ldots - |b_{n-1}|^2$$

$$= \| f(z) \|^2_B - | b_0 z^{n-1} + \ldots + b_{n-1} |^2$$

$$= \| R(0)^{*n} f(z) \|^2_B \, .$$

This proves the sufficiency part of the Theorem.

Necessity) Suppose $\| R(0)^{*n} f(z) \|_B \longrightarrow 0$ as $n \to \infty$ for every element $f(z)$ of $H(B)$. Let M be the set of elements $(f(z), g(z))$ of $\mathbb{D}(B)$ for which $\| T^{*n}(f(z), g(z)) \|_{\mathbb{D}(B)} \longrightarrow 0$ as $n \to \infty$. Because

$$\| T^{*n}((1-B(z)\bar{B}(0))c, \; \mathcal{J}^{*}(z\, \frac{-\bar{B}(0)}{z} c)) \|_{\mathbb{D}(B)} = \| R(0)^{*n} (1-B(z)\bar{B}(0))c \|_B \, ,$$

therefore $((1-B(z)\bar{B}(0))c, \frac{B^{*}(z)-\bar{B}(0)}{z} c)$ belongs to M for each vector c.

Since,

$$T^{*}(\frac{B(z)-B(0)}{z} c, (1-B^{*}(z)B(0))c) = -((1-B(z)\bar{B}(0))B(0)c, \frac{B^{*}(z)-\bar{B}(0)}{z} B(0)$$

belongs to M, so $(\frac{B(z)-B(0)}{z} c, (1-B^{*}(z)B(0))c)$ belongs to M for all

constants c in $\mathcal{S}$. If $(f(z),g(z))$ belongs to $\mathbb{D}(B)$, then

$$(1-T^*T)(f(z),g(z)) = \left((1-B(z)\overline{B}(0))\, f(0),\ \frac{B^*(z)-\overline{B}(0)}{z}\, f(0)\right),$$

and so the range of $I-T^*T$ is contained in $\mathcal{M}$. By computation, it follows that $T^{*n}T^n(f(z),g(z))$ belongs to $\mathcal{M}$ whenever $(f(z),g(z))$ belongs to $\mathcal{M}$. So $\mathcal{M}$ is invariant under T. Therefore $\mathcal{M}$ is a subspace of $\mathbb{D}(B)$ which reduces T and contains $\left((1-B(z)\overline{B}(0))\,c,\ \frac{B^*(z)-\overline{B}(0)}{z}\,c \right)$ and $\left(\frac{B(z)-B(0)}{z}\,c,\ (1-B^*(z)B(0))\,c\right)$ for every c in $\mathcal{S}$. It follows by the definition of T and T^* that $\mathcal{M}$ is dense in $\mathbb{D}(B)$. It is evident that $\mathcal{M}$ is closed and so coincides with $\mathbb{D}(B)$.

<u>COROLLARY:</u> The multiplication by $B^*(z)$ is isometric in $\mathcal{S}(z)$, provided $T^{*n} \longrightarrow 0$ in $\mathbb{D}(B)$ as $n \to \infty$.

<u>Proof</u>. Because $T^{*n} \to 0$ as $n\to\infty$ in $\mathbb{D}(B)$, so $\|(f(z),g(z))\|_{\mathbb{D}(B)} = \|g(z)\|$. By definition of $\mathbb{D}(B)$, $\|g(z)\|_{B^*} \leq \|(f(z),g(z))\|_{\mathbb{D}(B)}$, which means that $H(B)$ is contained isometrically in $\mathcal{S}(z)$. So by Theorem 4 ([6] ; p.349), the multiplication by $B^*(z)$ is a partial isometry. If $B^*(z)c=0$, then

$$\left((1-B(z)\overline{B}(0))c,\ \frac{B^*(z)-\overline{B}(0)}{z}\,c \right) = (c,0)$$

belongs to $\mathbb{D}(B)$. So $\|T^{*n}(c,0)\|_{\mathbb{D}(B)} = \|(c,0)\|_{\mathbb{D}(B)}$ holds for all $n=1,2\ldots$ Hence it follows by the hypothesis that c=0. Therefore the overlapping space $\mathbb{L}(\emptyset)$ of $H(B^*)$ has no non-zero polynomial. Hence the transformation $l(z) \longrightarrow \frac{l(z)-l(0}{z}$ in $\mathbb{L}(\emptyset)$ is isometric.

If $B^*(z)\ l(z)=0$, then $l(z)$ belongs to $\mathbb{L}(\phi)$ and $\|l(z)\|_{\mathbb{L}(\phi)} = \|l(z)\|$. But then $l(0)=0$ because

$$\|l(z)\|^2 = \left\|\frac{l(z)-l(0)}{z}\right\|^2 - |l(0)|^2 = \left\|\frac{l(z)-l(0)}{z}\right\|^2_{\mathbb{L}(\phi)}$$

and $\left\|\frac{l(z)-l(0)}{z}\right\|_{\mathbb{L}(\phi)} \geq \left\|\frac{l(z)-l(0)}{z}\right\|$. Therefore $l(z)$ vanishes identically. Hence the multiplication by $B^*(z)$ in $S(z)$ is an isometry.

4.1.3 The following Theorem is helpful in the investigation of $H(B^*)$.

THEOREM 3: Let $H(B)$ be a given space such that there is no non-zero element in $H(B^*)$ of the form $B^*(z)c$ for any c in S. Assume that the elements of the form $B(z)l(z)$ where $l(z)$ belongs to the over lapping space $\mathbb{L}(\phi)$ of $H(B)$, form a dense set in $H(B)$ and that $\|f(z)\|_{B^*} = \|f(z)\|$ for all elements $f(z)$ of $\mathbb{L}(\phi^*)$. Then the multiplication by $B^*(z)$ in $S(z)$ is an isometry.

Proof. By Theorem 11([6];p.355), $\mathbb{L}(\phi^*)$ is contained in $H(B^*)$ and the inclusion does not increase norms, i.e. $\|g(z)\|_{B^*} \leq \|g(z)\|_{\mathbb{L}(\phi^*)}$ for every element $g(z)$ of $\mathbb{L}(\phi^*)$. Moreover the identity

$$\bar{c}\ \tilde{l}(w) = \left\langle l(z), \frac{1}{2}\ \frac{\phi(z)-\phi(\bar{w})}{z-\bar{w}}\, c \right\rangle_{\mathbb{L}(\phi)}$$

$$= -\left\langle B(z)l(z), \frac{B(z)-B(\bar{w})}{z-\bar{w}}\, c \right\rangle_B$$

holds for all elements $l(z)$ of $\mathbb{L}(\phi)$, c on S when $|w|<1$. By the given hypothesis, it follows that transformation $f(z) \to \tilde{f}(z)$ is isometry on $H(B)$ into $H(B^*)$. Therefore in view of above identity, we have,

$$\| B(z)\tilde{l}(z) \|_B = \|\tilde{l}(z)\|_{B^*} = \|l(z)\|$$

for each element $l(z)$ of $\mathbb{L}(\emptyset)$. If $(f(z),g(z))$ belongs to $\mathbb{E}(\emptyset)$ the extension space of $\mathbb{L}(\emptyset)$, then $(B(z)f(z),-g(z))$ belongs to $\mathbb{D}(B)$ the extension of $H(B)$, and

$$\|(f(z),g(z))\|^2_{\mathbb{E}(\emptyset)} = \|f(z)\|^2 + \|(B(z)f(z),-g(z))\|^2_{\mathbb{D}(B)}$$

$$= \|f(z)\|^2 + \|(B(z)f(z),-\tilde{f}(z))\|^2_{\mathbb{D}(B)} + \|(0,\tilde{f}(z)-g(z))\|^2_{\mathbb{D}(B)}$$

$$= \|f(z)\|^2 + \|\tilde{f}(z)\|^2_{B^*} + \|\tilde{f}(z)-g(z)\|^2_{B^*}$$

$$= \|f(z)\|^2 + \|g(z)\|^2_{B^*}$$

$$= \|f(z)\|^2 + \|g(z)\|^2,$$

i.e. $\|(f(z),g(z))\|_{\mathbb{E}(\emptyset)} = \|(f(z),g(z))\|_{\mathbb{E}(1)}$

for every element $(f(z),g(z))$ of $\mathbb{E}(\emptyset)$. Therefore it follows that $\mathbb{L}(\emptyset)$ is contained isometrically in $S(z)$. So there is no non-zero space $\mathbb{L}(\psi)$ contained in $\mathbb{L}(\emptyset)$ and $\mathbb{L}(1-\emptyset)$ such that the inclusion do not increase norms. The Theorem now follows by using the result of Theorem 5 ([7]; p. 126) .

Remark: It is shown in Chapter 5 that under the hypothesis of Theorem, $B(z)=B(0)$.

4.1.4 We see in the following section that a space $H(B)$ which has a non-zero element of the form $B(z)c$ for some vector c, can be replaced by a space so that equality is attained in (1.2.4) for every element of the new space.

THEOREM 4: Let $H(B)$ be a given space such that (i) $H(B^*)$ does not contain any element of the form $B^*(z)c$ for any non-zero constant c, (ii) sum of the dimension of the closure of the range of $(1-R(0)^*R(0))$ in $H(B)$ with the dimension of the closed span of constants which satisfy (3.2.0), does not exceed the dimension of the coefficient space and (iii) range of $B(\alpha)$ for some $\alpha, |\alpha| < 1$, is dense in S. Then $B(z) = CA(z)$ such that equality is attained in (1.2.4) for every element of $H(A)$, and $H(A)$ coincides with the orthogonal complement of $H(C)$ in $H(B)$.

Proof. Because (i) holds, Theorem 13 ([6] ; p. 354) implies that the transformation $U: f(z) \to \tilde{f}(z)$ defined on $H(B)$ into $H(B^*)$ is isometric, where $\tilde{f}(w)$ is defined by (1.2.6). As $U: (zf(z) - B(z)\tilde{f}(0)) \to \frac{\tilde{f}(z)-\tilde{f}(0}{z}$ whenever $U: f(z) \to \tilde{f}(z)$, so the range M of U, is a closed subspace of $H(B^*)$ which posses $\frac{f(z)-f(0)}{z}$ alongwith $f(z)$ such that

$$\left\| \frac{f(z)-f(0}{z} \right\|_M^2 = \| f(z) \|_M^2 - | f(0) |^2 .$$

If $S(0): f(z) \to \frac{f(z)-f(0}{z}$ in M then $(1-S(0)S(0)^*) = U(1-R(0)^*R(0))U^*$, which implies that the dimension of the closure of the range of $1-S(0)S(0)^*$ in M, is equal to the dimension of the closure of the range of $1-R(0)^*R(0)$ in $H(B)$.

Consider the Hilbert space H_0 consisting of power series $f(z)$ such that $\frac{f(z)-f(0)}{z}$ belongs to $H(B)$ and

$$\operatorname{Sup} \left(| f(0)+g(0)|^2 + \left\| \frac{f(z)-f(0)}{z} + \frac{f(z)-g(0)}{z} \right\|_B^2 - \| g(z) \|_B^2 \right) < \infty$$

where the supremum is taken over all elements $g(z)$ of $H(B)$. The elements of H_0 are of the form $B(z)c$ where c is some constant, and $\|B(z)c\|_0 = |c|$ whenever $B(z)c \neq 0$. As the orthogonal complement of the set of elements of the form $B(z)\tilde{f}(0)$ in H_0 for each $f(z)$ of $H(B)$, is prescisely the set of constants contained in H_0, hence the dimension of the closed span of vectors orthogonal to the coefficients of elements of M is equal to the dimension of the closed span of constants which satisfy (3.2.0). Therefore, keeping Theorem 6([6]; p. 351) and hypothesis (ii) in the view,it follows that M is equal isometrically to some $H(A^*)$ where $A^*(z)$ is a power series with operator coefficients such that $H(A^*)$ does not contain any non-zero element of the form $A^*(z)c$ for any constant c. Since $H(A^*)$ is contained isometrically in $H(B^*)$, $B^*(z) = A^*(z)\, C^*(z)$ such that $H(C^*)$ exists.

Since $U: \dfrac{1-B(z)\overline{B}(w)}{1-z\overline{w}}\, c \rightarrow \dfrac{B^*(z)-\overline{B}(w)}{z-\overline{w}}\, c$ for every constant c,

the element

$$A^*(z)\,\frac{C^*(z)-\overline{C}(w)}{z-\overline{w}}\, c = \frac{B^*(z)-\overline{B}(w)}{z-\overline{w}}\, c - \frac{A^*(z)-\overline{A}(w)}{z-\overline{w}}\,\overline{C}(w)c$$

belongs to M for every constant c and complex number w, $|w|<1$. So, by Theorem 4F([6]; p. 350), $A^*(z)\,\dfrac{C^*(z)-\overline{C}(w)}{z-\overline{w}} = 0$ for each w, $|w|<1$, which means $\overline{B}(\alpha) = \overline{A}(\alpha)\,\overline{C}(\alpha) = \overline{A}(\alpha)\,\overline{C}(w)$, or $\overline{A}(\alpha)(\overline{C}(\alpha)-\overline{C}(w))=0$ for every α, $|\alpha|<1$. Therefore,by the arbitrariness of w, $(c(z)-C(\alpha))A(\alpha)=0$. As $B(\alpha)$ has dense

range for some $\alpha, |\alpha|<1$, $C(z)-C(w)=0$ which means that $C(z)=C$ is a constant. So $B(z)=CA(z)$.

Since $H(B)$ is the closed span of elements of form $\frac{1-B(z)\overline{B}(w)}{1-z\overline{w}}c$,

$$U: \frac{1-B(z)\overline{B}(w)}{1-z\overline{w}} \longrightarrow \frac{B^*(z)-\overline{B}(w)}{z-\overline{w}}c$$

and since $\frac{B^*(z)-\overline{B}(w)}{z-\overline{w}}c = \frac{A^*(z)-\overline{A}(w)}{z-\overline{w}}\overline{C}c$, hence $H(A^*)$ is the closed span of the elements $\frac{A^*(z)-\overline{A}(w)}{z-\overline{w}}c$, c in $\mathcal{S}$. As $A^*(z)c$ is no non-zero element in $H(A^*)$ for any non-zero c in $\mathcal{S}$,

$$\left\| \frac{1-A(z)\overline{A}(w)}{1-z\overline{w}} c \right\|_A = \left\| \frac{A^*(z)-\overline{A}(w)}{z-\overline{w}} c \right\|_{A^*}.$$

Therefore the transformation $f(z) \longrightarrow \tilde{f}(z)$ defined on $H(A)$ into $H(A^*)$ by (1.2.6), is isometric and has dense range in $H(A^*)$. It means that $f(z) \longrightarrow \tilde{f}(z)$ is an unitary transformation on $H(A)$ onto $H(A^*)$. Therefore $H(A)$ does not contain any non-zero element of the form $A(z)c$ for any c in $\mathcal{S}$, so the identity

$$\left\| \frac{f(z)-f(0)}{z} \right\|_A^2 = \|f(z)\|_A^2 - |f(0)|^2$$

holds for every element $f(z)$ of $H(A)$. If $f(z)$ is in $H(A)$, then $\hat{f}(z)$ defined by

$$\overline{c}\,\hat{f}(w) = \left\langle f(z), \frac{A(z)-A(\overline{w})}{z-\overline{w}} c \right\rangle_A$$

belongs to $H(A^*)$ and there is an unique element $g(z)$ of $H(B)$ such that $\hat{f}(z)=\tilde{g}(z)$ and $\|f(z)\|_A = \|\hat{f}(z)\|_{A^*} = \|\tilde{g}(z)\|_{B^*} = \|g(z)\|_B$. Moreover,

$$\overline{c}g(w) = \langle g(z), \frac{1-B(z)\overline{B}(w)}{1-z\overline{w}} c \rangle_B$$

$$= \langle \tilde{g}(z), \frac{B^*(z)-\overline{B}(w)}{z-\overline{w}} c \rangle_{B^*}$$

$$= \langle f(z), \frac{A^*(z)-\overline{A}(w)}{z-\overline{w}} \overline{C} c \rangle_{A^*}$$

$$= \overline{c}\, Cf(w)$$

for all c in $\mathcal{S}$ when $|w| < 1$. Therefore the transformation $f(z) \to Cf(z)$ is an isometry on $H(A)$ into $H(B)$. Obviously, the range is the orthogonal complement of $H(C)$ in $H(B)$. The Theorem is proved by observing the equality in (1.2.4) for every element of $H(C)$.

(4.1.5) <u>COROLLARY:</u> Assume $\mathcal{S}$ to be infinite dimensional. Let $H(B)$ be a given space such that there is no non-zero c in $\mathcal{S}$ such that $B^*(z)c$ belongs to $H(B)$. Then $B(z)=CA(z)$ such that equality is attained in (1.2.4) for every element of $H(A)$ and the transformation $f(z) \to Cf(z)$ maps $H(A)$ isometrically onto the orthogonal complement of $H(C)$ in $H(B)$.

<u>Proof:</u> In the proof of the Theorem, $\mathcal{M}$ is equal isometrically to some $H(A^*)$. This follows because the range of $1-S(0)S(0)^*$ is contained in $\mathcal{S}(z)$, hence its dimension can not exceed the dimension of coefficient space which is infinite. The corollary is obtained by revising the remaining arguements of the proof of the Theorem.

(4.1.6) The spectrum of the transformation $f(z) \to \frac{f(z)-f(0)}{z}$ in $H(B)$ spaces, is related with analytic continuation of $B(z)$ across the unit circle and the zeros of $B(z)$. We discuss this first before obtaining a factorization of $B(z)$. The following Lemma extends the

LEMMA 1: Let $H(B)$ be a given space. The elements of $H(B)$ have analytic continuation over the boundary of the unit circle, at λ, if and only if, $B(z)$ has analytic continuation at the point λ.

Proof: Assume that elements of $H(B)$ have analytic continuation to a point $z=\lambda$, $|\lambda|=1$. Then in particular, $\frac{B(z)-B(0)}{z}c$ can be continued analytically to $z=\lambda$, for each vector c. Therefore $B(z)$ has analytic extension to the point $z=\lambda$. For the converse, assume that $B(z)$ can be continued analytically to $z=\lambda$, $|\lambda|=1$. Since the denominator of $\frac{1-B(z)\overline{B}(w)}{1-z\overline{w}}c$ is analytic everywhere except at $z=\frac{1}{\overline{w}}$ and $(1-B(z)\overline{B}(w))c$ is analytic at λ, $|\lambda|=1$, therefore $\frac{1-B(z)\overline{B}(w)}{1-z\overline{w}}c$ has analytic continuation . Because such special elements form a dense set in $H(B)$, and uniform convergence of analytic functions is analytic, therefore the elements of $H(B)$ have analytic continuation at λ, $|\lambda|=1$. This proves the Lemma.

LEMMA 2: Let $H(B)$ be a given space such that $\overline{B}(w)$ has dense range for some w, $|w|<1$. If $\overline{\lambda}$ belongs to the resolvent set of $R(0)$, $|\lambda|<1$ and $(R(0)-\overline{\lambda})$ has closed range in $H(B)$ then $B(\lambda)$ has bounded operator inverse.

Proof: Assume that $\overline{\lambda}$ belongs to the resolvent set of $R(0)$, $|\lambda|<1$ and $(R(0)-\overline{\lambda})$ has closed range, or equivalently, λ belongs to the resolvent set of $R(0)^*$ and the range of $(R(0)^*-\lambda)$ is closed. Assume that $B(\lambda)c=0$ for some constant c. Since $\frac{B(z)-B(\lambda)}{z-\lambda}c$ belongs to $H(B)$, $\frac{B(z)c}{z-\lambda}$ belongs to $H(B)$ for some constant c. Then it is obtained by computation that $(R(0)^*-\lambda)\frac{B(z)c}{z-\lambda}=0$ which means that $B(z)c=0$.

Therefore $B(w)c=0$ for all complex number w, $|w|<1$ but by the hypothesis $\overline{B}(w)$ has dense range for some w, $|w|<1$, so $c=0$. It proves that $B(\lambda)$ is one-one. Since the range of $(R(0)^*-\lambda)$ is dense in $H(B)$, so by the hypothesis the range of $(R(0)^*-\lambda)$ coincides with $H(B)$. Therefore, if $g(z)$ belongs to $H(B)$ then there is an element $f(z)$ of $H(B)$ such that $(R(0)^*-\lambda)f(z)=g(z)$, i.e., $f(z) = \frac{g(z)-B(z)\tilde{f}(0)}{z-\lambda}$, where

$$\bar{c}\,\tilde{f}(0) = \langle f(z), \frac{B(z)-B(0)}{z} c \rangle_B .$$

Since $f(z)$ converges everywhere in the unit disc, $g(\lambda)=B(\lambda)f(0)$ which implies that the range of $B(\lambda)$ contains $f(\lambda)$ for every $f(z)$ in $H(B)$. If c be a constant such that $c=f(z)+B(z)g(z)$ is a decomposition of c in $S(z)$ with $f(z)$ in $H(B)$ then $c=f(\lambda)+B(\lambda)g(\lambda)$ which implies that the range of $B(\lambda)$ contains all constants. As it is already shown that $B(\lambda)$ is one-one, so $B(\lambda)$ has bounded operator inverse. This completes the proof of the Lemma.

<u>THEOREM 5</u>: Let $H(B)$ be a given space. The function $B(z)$ has analytic continuation to the unit circle, if and only if, the unit circle is contained in the resolvent set of $R(0)$.

<u>Proof</u>: By Lemma 1, elements of $H(B)$ and $H(B^*)$ have analytic continuation to the unit circle. If c is in S and $|w|<1$ then the identity

$$\bar{c}\, f(w) = \langle f(z), \frac{1-B(z)\overline{B}(w)}{1-z\bar{w}} c \rangle_B$$

holds for every element $f(z)$ of $H(B)$. As $f(z) \to f(w)$ is continuous for each number w of the unit circle, the representation holds for all w, $|w| = 1$. Because $\frac{1-B(z)\overline{B}(w)}{1-z\overline{w}}c$ belongs to $H(B)$ for each complex number w, $|w| = 1$, and $\frac{1-\overline{B}^*(z)B(\overline{w})}{1-z\overline{w}}d$ belongs to $H(B^*)$, so $\overline{B}(w)c \neq 0$ and $B(\overline{w})d \neq 0$ unless c and d are zero. In other words, $(R(0)-\overline{\lambda})$ has no non-zero element in its kernel when $|\lambda| = 1$. If $(R(0)^*-\lambda)f(z)=0$, where $f(z)$ is some element of $H(B)$ then $f(z) = \frac{B(z)\tilde{f}(0)}{z-\lambda}$ where $\tilde{f}(0)$ is same as given by (1.2.6). By the analyticity of $f(z)$, $B(\lambda)\tilde{f}(0)=0$ meaning thereby $\tilde{f}(0)=0$. Therefore there is no non-zero element in the kernel of $(R(0)^*-\lambda)$, or equivalently, the range of $(R(0)-\lambda)$ is dense in $H(B)$.

Let w be a complex number close to the unit disc but $|w| > 1$. Then

$$\|(R(0)-w)^{-1}\| = \frac{1}{|w|}\,\|(1-\tfrac{1}{w}R(0))^{-1}\| \leq \frac{1}{|w|-1}$$

and the power series

$$(R(0)-\lambda)^{-1} = (R(0)-w)^{-1} + (\lambda-w)\,(R(0)-w)^{-2}\,\Sigma\,(\lambda-w)^n(R(0)-w)^{-n}$$

converges if $|\lambda-w| < \|(R(0)-w)^{-1}\|^{-1}$. Therefore there exists a disc centred at w which does not contain any point of the spectrum. By the arbitrariness of w, it follows that the resolvent set of $R(0)$ contains entire unit circle.

Assume that $\overline{\lambda}$ belongs to the resolvent set of $R(0)$ and $|\lambda|=1$. Since resolvent set is open, there exists a neighbourhood of λ contained in the resolvent set. So $(R(0)-w)$ has bounded inverse for

each w in some neighbourhood of $\bar{\lambda}$. The transformation $J(0): f(z) \to f(0)$ is continuous defined on $H(B)$ into S. Since $\bar{\lambda} = \lambda^{-1}$,

$$J(0)\ (1-wR(0))^{-1} f(z) \to f(w)$$

is continuous, for each w in the neighbourhood of λ,

$$|f(w)| \leq \|f(z)\|_B \|J(0)\| \|(1-wR(0))^{-1}\| .$$

Therefore the elements of $H(B)$ have analytic continuation to $z=\lambda$. By the previous Lemma, $B(z)$ can be continued analytically to $z=\lambda$. What remains to prove is that $B(z)$ is bounded in some neighbourhood of λ. Since $\frac{B(z)-B(0)}{z}c$ belongs to $H(B)$ for each constant c in S, so $\frac{B(w)-B(0)}{z}c$ is bounded in some neighbourhood of λ. Hence $B(z)$ is also bounded in the same neighbourhood. This completes the proof of the Theorem.

<u>THEOREM 6</u>: Let $H(B)$ be a given space which has no element of the form $B(z)c$ for any non-zero c in S. Assume that $(1-B(0)\bar{B}(0))$ is a completely continuous operator and $B(w)$ has dense range for some w, $|w| < 1$. Then $B(z) = A(z)C(z)$ such that $A(z) = \prod_{i=0}^{n} (1-P_i+P_i z)U$ (for some unitary transformation U and projection operators $P_i, i=0,1,\ldots,n$), $H(C)$ contains no non-zero polynomial and $C(0)$ has its range equal to S. $C(\bar{w})$ has bounded inverse, whenever $w \neq 0$ $g(z)=R(w)^* f(z), f(z) \neq 0$ imply that $f(0)$ and $g(0)$ are linearly independent vectors.

<u>**Proof:**</u> If $H(B)$ contains no non-zero element, then $B(z) = B(0)$ and $\bar{B}(0)$ is isometric. The transformation $f(z) \longrightarrow \tilde{f}(z)$ defined by

$$\bar{c}\,\tilde{f}(\bar{w}) = \langle f(z), \frac{B^*(z)-\bar{B}(\bar{w})}{z-w}\, c \rangle_{B^*}$$

on $H(B^*)$ into $H(B)$ is isometric which implies that $H(B^*)$ is also the null space. It determines that $B(0)$ is an isometry. The theorem follows by taking $P_i=I$, $C(z)=I$ and $U=B(0)$.

Suppose $H(B)$ does contain a non-zero element. If it has no non-zero polynomial then $B(z)=C(z)$ and we only need to check that the range of $B(0)$ coincides with $\mathcal{S}$. Since $(1-\bar{B}(z)B(0))c$ belongs to $H(B)$ for each c in $\mathcal{S}$, hence $\bar{B}(0)c \neq 0$ unless $c=0$. It shows that $B(0)$ has dense range. By the hypothesis, $(1-B(0)\bar{B}(0))$ is completely continuous which implies that the range of $\bar{B}(0)$ is closed and therefore the range of $B(0)$ coincides with $\mathcal{S}$.

Therefore, in what follows we assume that $H(B)$ possesses a non-zero polynomial. Since $H(B)$ does not contain any non-zero element of the form $B(z)c$, the closure of polynomials in $H(B)$ is a closed subspace of $\mathcal{S}(z)$ isometrically contained in $H(B)$. The closure is invariant under the transformation $f(z) \longrightarrow \dfrac{f(z)-f(0)}{z}$ so it is isometrically equal to some $H(A)$ where $A(z)$ is a power series with operator coefficient such that the multiplication by $A(z)$ in $\mathcal{S}(z)$ is partially isometric. So $B(z)=A(z)C(z)$ for some power series $C(z)$ with operator coefficients such that the space $H(C)$ exists.

Consequently, the range of the multiplication by $A(z)$ in $\mathcal{S}(z)$, contains the range of the multiplication by $B(z)$ in $\mathcal{S}(z)$, and $A(w)$ has dense range in $\mathcal{S}$ for some w, $|w|<1$. A direct application of Theorem 16 ([5] ; p.354) follows that the multiplication by $A(z)$ in $\mathcal{S}(z)$ can be assumed to be isometric without changing $F(A)$. Since

$$0 \leq \bar{c}(1-A(0)\bar{A}(0))c = \langle (1-A(z)\bar{A}(0))c, (1-A(z)\bar{A}(0))c \rangle_A$$

$$= \langle (1-A(z)\bar{A}(0))c, (1-B(z)\bar{B}(0))c \rangle_B$$

$$\leq \|(1-A(z)\bar{A}(0))c\|_B \|(1-B(z)\bar{B}(0))c\|_B$$

or, $\bar{c}(1-A(0)\bar{A}(0))c \leq \bar{c}(1-B(0)\bar{B}(0))c.$

By the arbitrariness of constant c and complete continuity of $(1-B(0)\bar{B}(0))$, it follows that $(1-A(0)\bar{A}(0))$ is also completely continuous. Applying Theorem 17 ([6]; p.355) it is obtained that

$$A(z) = (1-P_0+P_0 z)\dots(1-P_n+P_n z)\, U,$$

where P_i, $i=0,1,\dots n$, are projection operators with finite dimensional ranges and U is an unitary operator.

Assume that $H(C)$ has a polynomial $p(z)$. Then the constant $p(0)$ belongs to $H(C)$. Consequently, $A(z)p(0)$ belongs to $H(B)$ and being a polynomial it also belongs to $H(A)$. Since $H(A)$ has no non-zero element of the form $A(z)p(0)$, $A(z)p(0)=0$. The isometry of the multiplication by $A(z)$ implies that $p(0)=0$. Repeating the same arguement, it follows that $p(z)=0$ and hence $H(C)$ does not contain any

non-zero polynomial. If c is a vector orthogonal to the range of $C(0)$, then for every element $f(z)$ of $\mathcal{S}(z)$,

$$0 = \langle C(z)f(z), c \rangle = \langle B(z)f(z), A(z)c \rangle .$$

It shows that $A(z)c$ belongs to $H(B)$. Being a polynomial it also belongs to $H(A)$ which implies that c is the null vector. It proves that the range of $C(0)$ is dense in $\mathcal{S}$.

If $J_A(0)$ denotes the transformation $f(z) \to f(0)$ defined on $H(A)$ into $\mathcal{S}$ then $J_A(0)J_A(0)^* = (1-A(0)\overline{A}(0))$ is completely continuous which implies that $J_A(0)$ and $J_A(0)^*$ are also completely continuous. Therefore,

$$J_A(w) = J_A(0)\ (1-wR_A(0))^{-1}: f(z) \to f(w)$$

is also completely continuous for every complex number w, $|w| < 1$. A composition of such transformations

$$J_A(\beta)\ J_A(\alpha)^*: c \to \frac{1-A(\alpha)\overline{A}(\beta)}{(1-\alpha\overline{\beta})}\, c$$

is also completely continuous. Since

$$A(\alpha)\ \frac{1-C(\alpha)\overline{C}(\beta)}{1-\alpha\overline{\beta}}\ \overline{A}(\beta) = \frac{1-B(\alpha)\overline{B}(\beta)}{1-\alpha\overline{\beta}} - \frac{1-A(\alpha)\overline{A}(\beta)}{1-\alpha\overline{\beta}}$$

whenever $|\alpha| < 1$ and $|\beta| < 1$, $A(\alpha)$ and $\overline{A}(\beta)$ have bounded inverses if $\alpha \neq 0$, $\beta \neq 0$, so $(1-C(\alpha)\overline{C}(\beta))$ is completely continuous for $0 < |\alpha| < 1$ and $0 < |\beta| < 1$. Therefore, by continuity, $(1-C(0)\overline{C}(0))$ is completely continuous. It implies that the ranges of $C(0)$ and $\overline{C}(0)$ are closed. Since the range of $C(0)$ is dense in $\mathcal{S}$, it coincides with $\mathcal{S}$.

Assume that $f(0)$ and $g(0)$ are linearly independent vectors whenever $g(z)=R(w)^*f(z)$ for some w, $0<|w|<1$, and $f(z)\neq 0$. We show that the range of $(R(0)-w)$ is dense in $H(B)$. Indeed, if $(R(0)^*-\bar{w})f(z)=0$ for some element $f(z)$ of $H(B)$ then

$$\langle f(z), \frac{g(z)-g(0)}{z} \rangle_B = \bar{w}\langle f(z),g(z)\rangle_B$$

for every element $g(z)$ of $H(B)$. Repeating the same, it is obtained that

$$\langle f(z), R(0)^n g(z)\rangle_B = \bar{w}^n\langle f(z),g(z)\rangle_B.$$

In particular, if $g(z)=(1-B(z)\bar{B}(0))c$ then the identity can be converted into identity

$$-\langle f(z), \frac{B(z)-B(w)}{z-w}\bar{B}(0)c\rangle_B = \sum_{n=0}^{\infty} \bar{w}^{2n+1}\langle f(z),(1-B(z)\bar{B}(0))c\rangle_B$$

or equivalently,

$$-\bar{c}B(0)\,\tilde{f}(\bar{w}) = \frac{\bar{w}}{1-\bar{w}^2}\,\bar{c}\,f(0).$$

By the arbitrariness of c, $B(0)\tilde{f}(\bar{w}) = -\dfrac{\bar{w}}{1-\bar{w}^2}f(0)$.

Therefore in the view of (1.2.6) and the hypothesis, it follows that $f(z)=0$. It proves that $(R(0)-w)$ has dense range in $H(B)$. Using the property that $H(B)$ has no non-zero element of the form $B(z)c$ for any non-zero vector c and applying the definitions of the concerning transformations, one obtains that the transformation

$$1-(1-wR(0)^*)^{-1}(R(0)^*-w)(R(0)-\bar{w})(1-\bar{w}R(0))^{-1}$$

$$=(1-|w|^2)\,J_B(w)^*J_B(w).$$

is completely continuous for all complex number w, $0<|w|<1$ which implies that the transformation

$$\{ (1-wR(0)^*)^{-1} (R(0)^*-\overline{w}) (R(0)-w) (1-\overline{w}R(0)^*)^{-1} \}$$

has closed range in $H(B)$. It leads to conclude that $(R(0)-w)$ has closed range in $H(B)$ whenever $0<|w|<1$. Therefore by the proof of Lemma 2, the range of $B(\overline{w})$ is dense in S. Since

$$J_B(\overline{w})\, J_B(w)^* = \frac{1-B(\overline{w})\overline{B}(\overline{w})}{1-|w|^2}$$

is completely continuous, the range of $B(\overline{w})$ is closed and hence coincides with S. By definition of $A(z)$, $A(\overline{w})$ has operator inverse so the range of $C(\overline{w})$ coincides with S . This completes the proof of the Theorem.

(4.1.7) The following Theorem is an extension of Theorem 7 ([7] ; p. 127). The result of the Theorem can be used in approximation of $H(B)$ spaces with finite dimensional spaces. It also helps to extend the result of Lemma 10 ([5] ; p. 71) on similar lines.

<u>THEOREM 7</u>: Let $A(z)$ be a power series with operator coefficients and $B(0)$be a normal operator, such that the multiplication by $A^*(z)$ is isometric in $S(z)$, there is no non-zero element $f(z)$ in $S(z)$ such that $zf(z) = \overline{B}(0)g(z)$ where $f(z)=h(z) + A^*(z)g(z)$, with $h(z)$ in $H(A^*)$ and $g(z)$ in $S(z)$, is the minimal decompositions of $f(z)$.

Then there exists a non-null space $H(B)$ for

An element $f(z)$ belongs to $H(B^*)$, if and only if, $(1-zB(0)A^*(z))f(z)$ belongs to $H(A^*)$ and

$$\|f(z)\|^2_{B^*} = \|B(0)f(z)\|^2 + \|(1-zB(0)A^*(z))f(z)\|^2_{A^*}.$$

Proof: To show that $H(B)$ exists, it is sufficient to prove that the multiplication by $B^*(z)$ is isometric in $S(z)$. Firstly, we show that the multiplication by $B^*(z)$ is isometric on the closure of the range of multiplication by $(1-zA^*(z)B(0))$ in $S(z)$. If $f(z)$ is in $S(z)$ then

$$B^*(z)\ (1-zA^*(z)B(0))f(z) = (\overline{B}(0)-zA^*(z))f(z).$$

So we need to check that the identity

$$\langle (1-zA^*(z)B(0))f(z),\ (1-zA^*(z)B(0))g(z) \rangle$$

$$= \langle (\overline{B}(0)-zA^*(z))f(z), (\overline{B}(0)-zA^*(z))g(z) \rangle$$

holds for all elements $f(z)$ and $g(z)$ of $S(z)$. By linearlity and continuity of the innerproduct, it is enough to establish the identity when $f(z)=z^n a$ and $g(z)=z^m b$ for all non-negative integers m,n and vectors a,b. Since the multiplication by z is isometric in $S(z)$, m is assumed to be zero. If $n=0$, the identity reduces to

$$\overline{b}a + \overline{b}\,\overline{B}(0)B(0)a = \overline{b}a + \overline{b}\,B(0)\overline{B}(0)a$$

which is true, because $B(0)$ is a normal operator. For $n \geq 1$, both sides of the identity vanish together.

If $f(z)$ is orthogonal to the closure of the multiplication by

$$(4.1.8) \qquad 0 = \langle f(z),\ (1-zA^*(z)B(0))z^n c \rangle$$

for all vectors c and integers $n \geq 0$. If $f(z)=h(z) + A^*(z)g(z)$ is the minimal decomposition of $f(z)$ in $\mathcal{S}(z)$ then the minimal decomposition of $\frac{f(z)-f(0)}{z}$ is given by

$$(4.1.9) \quad \frac{f(z)-f(0)}{z} = \left\{ \frac{h(z)-h(0)}{z} + \frac{A^*(z)-\bar{A}(0)}{z} g(0) \right\} + A^*(z) \frac{g(z)-g(0)}{z} ,$$

where the element in { } belongs to $H(A^*)$. If $f_0(z)=f(z)$,

$$f_{n+1}(z) = \frac{f_n(z)-f_n(0)}{z}, \quad g_0(z)= g(z) \text{ and } g_{n+1}(z) = \frac{g_n(z)-g_n(0)}{z},$$

$n=0,1,2\ldots$, the identity (4.1.8) together with (4.1.9) gives

$$\begin{aligned} \bar{c}\, a_n &= \langle f_{n+1}(z),\ A^*(z)B(0)c \rangle \\ &= \langle g_{n+1}(z),\ B(0)c \rangle \\ &= \bar{c}\, \bar{B}(0) b_{n+1} , \end{aligned}$$

where $g(z) = \Sigma b_n z^n$. By the arbitrariness of c and n, $zf(z)=\bar{B}(0)g(z)$. It implies (by the hypothesis) that $f(z)=0$. It leads to conclude that the multiplication by $B^*(z)$ is an isometry in $\mathcal{S}(z)$. Hence $B(z)$ converges to a function bounded by 1 in the unit disc. Consequently the space $H(B)$ exists.

For an element $f(z)$ of $\mathcal{S}(z)$ to be in $H(B^*)$, it is necessary and sufficient that $f(z)$ is orthogonal to the range of multiplication by $B^*(z)$. As the range of multiplication by $(1-zA^*(z)B(0))$ is dense in $\mathcal{S}(z)$, so the range of the multiplication by $B^*(z)$ is contained

in the closure of the range of multiplication by $(\overline{B}(0)-zA^*(z))$ in $\mathcal{S}(z)$. Therefore, $f(z)$ in $\mathcal{S}(z)$ belongs to $H(B^*)$, if and only if,

$$\langle f(z),\ (\overline{B}(0) - zA^*(z))z^n c \rangle = 0$$

for all vectors c and integers $n \geq 0$. If $f(z) = \Sigma a_n z^n$ then

$$\overline{c}\ B(0)\ a_n = \langle f_{n+1}(z),\ A^*(z)c \rangle .$$

Let $f(z) = h(z) + A^*(z)g(z)$ be the minimal decomposition of $f(z)$ with $h(z)$ in $H(A^*)$ then

$$\overline{c}\ B(0)\ a_n = \langle g_{n+1}(z),\ c \rangle$$

$$= \overline{c}\ b_{n+1} .$$

So $zB(0)\ f(z) = g(z)$. Hence $f(z)$ belongs to $H(B^*)$, if and only if, $(1-zA^*(z)B(0))f(z)$ belongs to $H(A^*)$ and

$$\| f(z)\|^2_{B^*} = \| B(0)\ f(z)\|^2 + \| (1-zA^*(z)B(0))f(z)\|^2_{A^*} .$$

This completes the proof.

4.2 A fundamental theorem of complex analysis states that the zeros of a nonzero analytic function are isolated and have no limit point in the region of analyticity. Various analogues of zeros are available for an operator-valued function $F(z)$. One may mean a point w to be a zero of $F(z)$ if, the inverse of $F(w)$ does not exist, the inverse is not defined densely, or the inverse is unbounded. One may also classify the zeros by considering the spectrum of $F(w)$. We call a point w to be a zero of $F(z)$ if $F(w)$ fails to have everywhere defined bounded inverse. In general if the coefficient space is of infinite dimension, such points are not isolated. For this reason the complex-valued analytic function theory is essentially different from the theory of operator-valued analytic functions, apart from non-commutativity of multiplication. Therefore it is of interest to know when the analogy between the two theories is sustained. In this section we study Blaschke product theory on the lines adopted by de Branges and Rovnyak for Weierstrass factorization, Theorem 20 ([6] ; p. 390). In the process, we use an extension of the Phragmén-Lindelöf principle for operator-valued functions given in Theorem 3.13.6 ([14]; p. 103).

Phragmén-Lindelöf Principle. Let $F(z)$ be an analytic function in the upper half plane such that $\|F(z)\|$ has a continuous extension to the closed half plane and is bounded by 1 on the real axis. If

$$\liminf_{r\to\infty} \frac{1}{r} \int_0^{\pi} \log^+ \| F(re^{i\theta})\| \sin\theta \, d\theta = 0 ,$$

where $\log^{+}\alpha = \max(\log\alpha, 0)$, then $F(z)$ is bounded by 1 in the upperhalf plan.

Definition An operator-valued analytic function $F(z)$ in the upper half plane is said to be of bounded type if there exists a complex-valued analytic function $\chi(z)$ such that $0 < |\chi(z)| \leq 1$ and $\|\chi(z)F(z)\| \leq 1$ whenever $\operatorname{Im} z > 0$.

(4.2.1) The following Theorem gives the usual form of Blastchke product.

THEOREM 8: Let $\{Q_n\}$ be a sequence of projection operators and $\{z_n = x_n + iy_n\}$ be a sequence of complex numbers such that $y_n > 0$ for $n \geq 1$ and $\Sigma \dfrac{y_n}{(x_n^2 + y_n^2)} < \infty$. Then the product

$$B(z) = \lim_{n\to\infty} \prod_{k=1}^{n} \left(I - Q_k \frac{z}{\bar{z}_k} \right)^{-1} \left(I - Q_k \frac{z}{z_k} \right) \tag{4.2.2}$$

converges in the operator norm uniformly on every bounded set which lies at positive distance from $\{\bar{z}_n\}$. $B(z)$ is analytic and bounded by 1 in the upper half plane and has operator inverse at each point in the upper half plane.

Proof: If we define,

$$\frac{1}{\rho(z)} = \sup_{k\geq 1} \left\| \left(\frac{I}{z} - Q_k \frac{1}{\bar{z}_k} \right)^{-1} \right\|,$$

then $0 < \rho(z) < \infty$ on every bounded set which lies at some positive distance from $\bar{z}_k$, $k \geq 1$. On computing,

$$(4.2.3) \quad \left\| \left(I-Q_k\frac{z}{\bar{z}_k}\right)^{-1} \left(I-Q_k\frac{z}{z_k}\right)\right\| \leq 1 + \left\| \left(I-Q_k\frac{z}{\bar{z}_k}\right)^{-1} \left(I-Q_k\frac{z}{z_k}\right)-I\right\|$$

$$\leq 1 + \frac{2\,y_k}{(x_k^2+y_k^2)} \left\| \left(\frac{I}{z} - Q_k\frac{1}{\bar{z}_k}\right)^{-1}\right\|$$

$$\leq \exp\left(\frac{2}{\rho(z)}\,\frac{y_k}{(x_k^2+y_k^2)}\right).$$

If we denote

$$B_n(z) = \prod_{k=1}^{n} \left(I-Q_k\frac{z}{\bar{z}_k}\right)^{-1} \left(I-Q_k\frac{z}{z_k}\right)$$

and make use of the norm inequality for operators

$$\|AB\| \leq 1 + \|AB-I\| \leq (1+\|A-I\|)\,(1+\|B-I\|)$$

then we get,

$$\|B_n(z)\| \leq \prod_{k=1}^{n}\left(1 + \left\|\left(I-Q_k\frac{z}{\bar{z}_k}\right)^{-1}\left(I-Q_k\frac{z}{z_k}\right) - I\right\|\right)$$

$$\leq \exp\left(\frac{2}{\rho(z)} \sum_{k=1}^{n} \frac{y_k}{x_k^2+y_k^2}\right).$$

Therefore,

$$\|B_n(z)-B_m(z)\| \leq \|B_m(z)\| \left(1 + \left\|\prod_{k=m+1}^{n} \left(I-Q_k\frac{z}{\bar{z}_k}\right)^{-1}\left(I-Q_k\frac{z}{z_k}\right) - I\right\|\right)$$

$$\leq \|B_m(z)\| \prod_{k=m+1}^{n}\left(1+\left\|\left(I-Q_k\frac{z}{\bar{z}_k}\right)^{-1}\left(I-Q_k\frac{z}{z_k}\right) - I\right\|\right)$$

As the sequence $\frac{2}{\rho(z)} \sum_{1}^{n} \frac{y_k}{x_k^2 + y_k^2}$ converges uniformly on every bounded set which lies at some positive distance from the points $\bar{z}_n$ for $n \geq 1$, it follows that $B_n(z) \longrightarrow B(z)$ as $n \to \infty$ in the operator norm uniformly, for z in any bounded set which lies at some positive distance from the points $\bar{z}_n$ for $n \geq 1$. The analyticity of $B(z)$ follows from the fact that $B(z)$ is uniform limit of analytic functions $B_n(z)$ in the upper half plane. If $z=x+iy$ then

$$(4.2.4) \qquad \left\| \left(I-Q_k \frac{z}{\bar{z}_k}\right)^{-1} \left(I-Q_k \frac{z}{z_k}\right) \right\| = \left\| \{(x_k-xQ_k)^2+(y_k-yQ_k)^2\} \{(x_k-xQ_k)^2+(y_k+yQ_k)^2\}^{-1} \right\|$$

$$= \left\| I-4yy_kQ_k \{(x_k-Q_kx)^2+(y_k+Q_ky)^2\}^{-1} \right\|$$

$$= \left\| I-4 \frac{yy_k}{x_k^2+y_k^2} Q_k \left\{ I- \frac{2xx_k-2yy_k-x^2-y^2}{x_k^2+y_k^2} Q_k \right\}^{-1} \right\|$$

Since $(x-x_k)^2+(y-y_k)^2 > 0$ for all z such that $\operatorname{Im} z > 0$, the expansion in Neumann series is permissible.

$$\left\| \left(I-Q_k \frac{z}{\bar{z}_k}\right)^{-1} \left(I-Q_k \frac{z}{z_k}\right) \right\| = \left\| I- \frac{4yy_k}{x_k^2+y_k^2} Q_k \left\{ 1+ \frac{2xx_k-2yy_k-x^2-y^2}{x_k^2+y_k^2} Q_k + \dots \right\} \right\|$$

$$= \left\| I- \frac{4yy_k}{x_k^2+y_k^2} \left\{ 1+ \frac{2xx_k-2yy_k-x^2-y^2}{x_k^2+y_k^2} + \dots \right\} Q_k \right\|$$

$$= \left\| I-4yy_k \{(x-x_k)^2+(y-y_k)^2\}^{-1} Q_k \right\| .$$

Since the inequality $\|I-\alpha Q_k\| \leq 1$ is valid for all α such that $0 \leq \alpha \leq 1$, so

$$\| B_n(z) \|^2 \leq \prod_{k=1}^{n} \| (I-Q_k \frac{z}{\bar{z}_k})^{-1} (I-Q_k \frac{z}{z_k}) \|^2$$

$$\leq \prod_{k=1}^{n} \| I-4yy_k \{(x-x_k)^2+(y+y_k)^2\} Q_k \|^2$$

$$\leq 1.$$

Because $B_n(z) \longrightarrow B(z)$ as $n \to \infty$ in operator norm, it follows that $\| B(z) \| \leq 1$ for all z in the upper half plane.

As,
$$(I-Q_k \frac{z}{\bar{z}_k})^{-1} = (z-\bar{z}_k)^{-1} (z-\bar{z}_k-Q_k z)$$

is well defined everywhere except for the point $z=\bar{z}_k$, so $B_n(z)$ assumes invertible values in the upper half plane. Therefore for each z in the upper half plane and sufficiently large n, $\|B_n(z)^{-1}B(z)-I\|<1$, which implies that $B_n(z)^{-1}B(z)$ assumes invertible values in the upper half plane. The last part of the Theorem follows by observing that $B(z)=B_n(z)B_n(z)^{-1}B(z)$ is invertible in the upper half plane. This completes the proof of the Theorem.

The following Theorem shows that the zeros of operator-valued function of bounded type are zeros of some Blaschke product.

THEOREM 9: Let $F(z)$ be an operator-valued function of bounded type in the upper half plane. Assume that there is no limit point $z=x+iy$ of zeros of $F(z)$ such that either $y>0$ or if $y=0$ then $x=0$. If the range of $F(w)$

for each zero w, is non-dense then $F(z)=B(z)G(z)$, where $B(z)$ is a Blaschke product of the form (4.2.2) and $G(z)$ is some analytic function of bounded type in the upper half plane, possessing operator inverse everywhere in the upper half plane.

Proof: Since $F(z)$ is of bounded type in the upper half plane, there exists an analytic function $\chi(z)$ such that $0 < |\chi(z)| \leq 1$ and $\|\chi(z)F(z)\| \leq 1$ in the upper half plane. If $F(z)$ is invertible at every point in the upper half plane, then the Theorem follows by taking $B(z)= 1$ and $G(z)=F(z)$. Assume that z_1 is the point nearest the origin such that $F(z_1)$ has no operator inverse. (Selection of such point is possible as the origin is not limit point of zeros of $F(z)$). Since the range of $F(z_1)$ is not dense, there exists a non-zero projection operator Q_1 whose range is the orthogonal complement of the range of $F(z_1)$.

Since $\left(I-Q_1 \frac{z}{\bar{z}_1}\right)\left(I-Q_1 \frac{z}{z_1}\right)^{-1}$ is an analytic function in the upper half plane except $z=z_1$ and $Q_1F_1(z_1)=0$, where $F(z)=F_1(z)$, hence

$$F_2(z) =\left(\left(I-Q_1 \frac{z}{\bar{z}_1}\right) \left(I-Q_1 \frac{z}{z_1} \right)^{-1}\right)F(z)$$

$$= \left(I-Q_1 \frac{z}{\bar{z}_1}\right) \left(F_1(z_1)+ (z-z_1-Q_1 z) \frac{F_1(z)-F_1(z_1)}{z-z_1}\right)$$

is analytic in the upper half plane. Inductively, let $F_n(z)$ be the function which fails to have operator inverse at z_n. Define

$$F_{n+1}(z) = \left(I-Q_n \frac{z}{\bar{z}_n}\right) \left(I-Q_n \frac{z}{z_n}\right)^{-1} F_n(z) , \tag{4.2.5}$$

where Q_n is the projection operator whose range is the orthogonal complement of the range of $F_n(z_n)$. If $F_{n+1}(z)$ has no zeros, then the theorem follows by taking $G(z)=F_{n+1}(z)$. So we consider the case when the process is continued. Let $F_n(z)$ be the sequence of functions obtained from (4.2.5). It is shown by induction that $\chi(z)F_n(z)$ is bounded by 1 in the upper half plane. For n=1, it is obviously true, so assume that $\chi(z)F_n(z)$ is bounded by 1. On following the lines adopted for (4.2.4), it is seen that

$$\left\| (I-Q_n \frac{z}{z_n}) (I-Q_n \frac{z}{z_n})^{-1} \right\| = \left\| I+ \frac{4yy_n}{x_n^2+y_n^2} Q_n (I- \frac{2xx_n+2yy_n-x^2-y^2}{x_n^2+y_n^2} Q_n)^{-1} \right\|$$

If $z = z_n$, then $(x-x_n)^2+(y-y_n)^2 > 0$. Therefore by expanding in Nuemann series

$$\left\|(I-Q_n \frac{z}{\bar{z}_n})(I-Q_n \frac{z}{z_n})^{-1} \right\|^2= \left\| I+ \frac{4yy_n}{x_n^2+y_n^2} Q_n(I+ \frac{2xx_n+2yy_n-x^2-y^2}{x_n^2+y_n^2} Q_n+\dots)\right\|$$

$$= \left\| I + \frac{4yy_n}{(x-x_n)^2+(y-y_n)^2} Q_n \right\| .$$

Since $\| I+ \alpha Q \|= (1+ \alpha)$, for $\alpha \geq 0$ and projection operator Q, hence

$$\left\| (I-Q_n \frac{z}{\bar{z}_n}) (I-Q_n \frac{z}{z_n})^{-1} \right\|^2 = 1+ \frac{4yy_n}{(x-x_n)^2+(y-y_n)^2} . \tag{4.2.6}$$

By definition

$$(4.2.7) \qquad \chi(z)F_{n+1}(z) = \left((I-Q_n \frac{z}{\bar{z}_n}) (I-Q_n \frac{z}{z_n})^{-1} \right) F_n(z)\, \chi(z).$$

In view of (4.2.6), it follows that the first factor in the right of (4.2.7) is bounded on every bounded set which lies at some positive distance from z_n. Since $\chi(z)F_{n+1}(z)$ is continuous at z_n, it is bounded in every bounded neighbourhood of z_n. Thus $\chi(z)F_{n+1}(z)$ is bounded in the upper half plane. We estimate the bound of $\chi(z)F_{n+1}(z)$ on each line $y=h>0$. By (4.2.6),

$$\max_x \| (I-Q_n \frac{z}{\bar{z}_n}) (I-Q_n \frac{z}{z_n})^{-1} \| = 1+ \frac{4yy_n}{(y-y_n)^2} = \frac{(h+y_n)}{|h-y_n|} \; .$$

An application of Phragmén-Lindelöf principle implies that $\chi(z)F_{n+1}(z)$ is bounded by $\frac{h+y_n}{|h-y_n|}$ in the half plane $y \geq h$. By the arbitrariness of h, it follows that $\chi(z)F_{n+1}(z)$ is bounded by 1 in the upper half plane. Writing $F(z)=B_n(z)F_{n+1}(z)$, where

$$B_n(z) = \prod_{k=1}^{n} (I-Q_k \frac{z}{\bar{z}_k})^{-1} (I-Q_k \frac{z}{z_k})$$

and noting that $\chi(z)\, F_{n+1}(z)$ is bounded by 1, we get,

$$\| \chi(z)F(z) \|^2 \leq \| B_n(z) \|^2 \leq \prod_{k=1}^{n} \| (I-Q_k \frac{z}{\bar{z}_k})^{-1} (I-Q_k \frac{z}{z_k}) \|^2$$

$$\leq \prod_{k=1}^{n} \| I - \frac{4yy_k}{(x-x_k)^2+(y+y_k)^2} Q_k \|$$

$$\leq \prod_{k=1}^{n} \left(1 + \frac{4yy_k}{(x-x_k)^2 + (y+y_k)^2}\right) .$$

Since n is arbitrary,

$$\| F(z)Q(z) \|^{-2} \geq \prod_{k=1}^{\infty} \left(1+ \frac{4yy_k}{(x-x_k)^2 + (y+y_k)^2}\right)^{-1}$$

$$\geq \prod_{k=1}^{\infty} \left(1- \frac{4yy_k}{(x-x_k)^2 + (y+y_k)^2}\right) .$$

Because $\| \chi(z)F(z) \|$ is not identically zero, the product converges for each z except for $z=z_k$. So

$$\sum_{1}^{\infty} \frac{yy_k}{x_k^2 + (y+y_k)^2} < \infty ,$$

for all $y \neq y_k$. Since (x_k+iy_k) has no subsequence converging to origin,

$$\sum_{1}^{\infty} \frac{y_k}{x_k^2 + y_k^2} < \infty .$$

So by Theorem 8, it follows that $B_n(z) \longrightarrow B(z)$ as $n \to \infty$ in the operator norm uniformly on every bounded set in the upper half plane and B(z) is analytic, bounded by 1 in the upper half plane. Thus $F(z)=B(z)G(z)$, where $G(z)= \lim_{n\to\infty} F_{n+1}(z)$ uniformly on every bounded set in the upper half plane. Since $\| \chi(z)F_{n+1}(z)\| \leq 1$ for all $n \geq 1$, it follows that G(z) is an analytic function such that $\| \chi(z)G(z)\| \leq 1$ in the upper half plane. By the construction, it is concluded that

the zeros of $F(z)$ coincide with those of $B(z)$. Since $F_{n+1}(z)$ and $B_n(z)^{-1}B(z)$ have operator inverse at every point in the semi disc of radius $|z_{n+1}|$ and

$$F_{n+1}(z) = B_n(z)^{-1}B(z)G(z),$$

hence $G(z)$ has operator inverse at every point in the semi-disc of radius $|z_{n+1}|$. The arbitrariness of n implies that $G(z)$ has operator inverse at every point in the upper half plane. Hence the Theorem.

<u>COROLLARY</u>: Let $F(z)$ be an operator-valued function of bounded type in the upper half plane and its extension be continuous in the closed half plane. Assume that $I-F(z)$ takes completely continuous values in the upper half plane. If $F(0)$ has dense range then $F(z)=B(z)G(z)$, where $B(z)$ is a Blaschke product and $G(z)$ is an operator-valued analytic function of bounded type in the upper half plane assuming invertible values such that $I-G(z)$ is completely continuous throughout the upper half plane.

<u>Proof</u>: If $f(z) = F\left(\frac{i-z}{i+z}\right)$, then $f(z)$ is a function of bounded type in the unit disc and so there exists a complex-valued analytic bounded function $\phi(z)$ such that $\|\phi(z)f(z)\| \leq 1$ and $I-\phi(z)f(z)$ is completely continuous whenever $|z| \leq 1$. So by Theorem 18 ([6]; p. 355), zeros of $\phi(z)f(z)$ are isolated which means that the zeros of $f(z)$ are isolated. It follows by the previous Theorem that $F(z)=B(z)G(z)$, where

$$B(z) = \lim_{n\to\infty} \prod_{n=1}^{n} \left(I-Q_k \frac{z}{\bar{z}_k}\right)^{-1} \left(I-Q_k \frac{z}{z_k}\right).$$

Since $I-F(z)$ takes completely continuous, hence by the construction , $I-B_n(z)$ takes completely continuous values in the upper half plane. As

$$I - F_{n+1}(z) = I - B_n(z)^{-1}F(z)$$

$$= B_n(z)^{-1} \quad (B_n(z)-I)+(I-F(z)) \quad ,$$

so $I-F_{n+1}(z)$ assumes completely continuous values. Because $I-G(z) = \lim_{n\to\infty} \quad I-F_n(z)$, therefore $I-G(z)$ also assumes completely continuous values.

CHAPTER V

FACTORIZATION AND INVARIANT SUBSPACES

Summary of the Chapter.

We study in this chapter, the problem described in section 1.5 for linear operators, and obtain a number of factorization theorems in the process. Theorem 1 of the chapter improves the result of Sz.-Nagy and Foias [23] and the hypothesis of the Theorem 6 ([7] ;p. 126), by showing that a contraction transformation T in a Hilbert space, has a non-trivial invariant subspace if either of the sequences $\{T^n\}$ or $\{T^{*n}\}$ does not converge strongly to zero, as $n \to \infty$. The proof is based on $\mathbb{L}(\emptyset)$ spaces theory and the duality between H(B) and H(B^*). In Theorem 2, we show that T (a contraction in a Hilbert space) has a non-trivial invariant subspace if either $(1-T^*T)$ or $(1-TT^*)$ is completely continuous. The proof depends on an approximation of a given space H(B) with a sequence of finite dimensional spaces H(B_n) and factorizations of each $B_n(z)$. In the end, we conclude that the problem can be reduced to a (non-trivial) factorization of B(z), (an operator-valued analytic function bounded by 1), such that the multiplications by B(z) and $B^*(z)$ are isometric transformations in $\mathcal{S}(z)$, the ranges of B(w) and $\overline{B}(w)$ are dense in $\mathcal{S}$, for all w, $|w| < 1$; $\mathcal{M}$(B) H(B), H(B^*) and $\mathcal{M}$(B^*) are mutually disjoint.

LEMMA 1. Let $\mathbb{L}(\emptyset)$, $\mathbb{L}(\theta)$ and $\mathbb{L}(\psi)$ be given spaces such $\emptyset(z) = \theta(z) + \psi(z)$. If $\mathbb{L} = \mathbb{L}(\theta) \cap \mathbb{L}(\psi)$, then $\mathbb{L} = \mathbb{L}(\chi)$ for some $\chi(z)$. $\mathbb{L}$ is the null space if, and only if, $\mathbb{L}(\theta)$ is contained isometrically in $\mathbb{L}(\emptyset)$. The

transformation $f(z) \longrightarrow \frac{f(z)-f(0)}{z}$ in $\mathbb{L}(\chi)$ is unitary, if for every polynomial $g(z)$ in $\mathbb{L}(\theta)$, $\|g(z)\|_{\mathbb{L}(\theta)} = \|g(z)\|_{\mathbb{L}(\phi)}$.

<u>Proof</u>. Because $\phi(z) = \theta(z) + \psi(z)$, the spaces $\mathbb{L}(\theta)$ and $\mathbb{L}(\psi)$ are contained in $\mathbb{L}(\phi)$ and their inclusions in $\mathbb{L}(\phi)$ do not increase norms. Let $\mathbb{E}(\phi)$, $\mathbb{E}(\theta)$ and $\mathbb{E}(\psi)$ be their extension spaces. Let $\mathbb{E} = \mathbb{E}(\theta) \cap \mathbb{E}(\psi)$. Define $\mathbb{E}$-norm on $\mathbb{E}$ by

$$(5.1.1) \qquad \|(f(z),g(z))\|^2_{\mathbb{E}} = \|(f(z),g(z)\|^2_{\mathbb{E}(\theta)} + \|(f(z),g(z)\|^2_{\mathbb{E}(\psi)} - \|(f(z),g(z)\|^2_{\mathbb{E}(\phi)} .$$

Then $\mathbb{E}$ is a Hilbert space in $\mathbb{E}$-norm and contains $\left(\frac{f(z)-f(0)}{z}, zg(z)+f(0)\right)$ and $\left(zf(z) + g(0), \frac{g(z)-g(0)}{z}\right)$ along with $(f(z),g(z))$. It follows from Theorem 1 ([9]; p. 166) that $\mathbb{E} = \mathbb{E}(\chi)$ for some power series $\chi(z)$ with operator coefficients such that Re $\chi(w) \geq 0$ whenever $|w| < 1$. Moreover, $\mathbb{E}(\chi)$ is included in $\mathbb{E}(\theta)$ and $\mathbb{E}(\psi)$ and the inclusions do not increase norms.

It follows from Theorem 3 ([9]; p. 166) that every element $h(z)$ in $\mathbb{L}(\phi)$ has a unique minimal decomposition $h(z)=f(z)+g(z)$, with $f(z)$ in $\mathbb{L}(\theta)$ and $g(z)$ in $\mathbb{L}(\psi)$ such that

$$\|h(z)\|^2_{\mathbb{L}(\phi)} = \|f(z)\|^2_{\mathbb{L}(\theta)} + \|g(z)\|^2_{\mathbb{L}(\psi)} .$$

It is also deduced that if $h_k(z) = f_k(z)+g_k(z)$ be a decomposition of $h_k(z)$ in $\mathbb{L}(\phi)$, $f_k(z)$ in $\mathbb{L}(\theta)$ and $g_k(z)$ in $\mathbb{L}(\psi)$, $k=1,2$ and either of the decompositions is minimal then

Now, let $\mathbb{L}(\theta)$ be contained in $\mathbb{L}(\phi)$ isometrically. If $f(z)$ is in $\mathbb{L}$, then $f(z) = f(z) + 0$ is the minimal decomposition of $f(z)$ in $\mathbb{L}(\phi)$ with $f(z)$ in $\mathbb{L}(\theta)$ and 0 in $\mathbb{L}(\psi)$. By (5.1.2),

$$\langle f(z), 0 \rangle_{\mathbb{L}(\phi)} = \langle f(z), f(z) \rangle_{\mathbb{L}(\theta)} + \langle 0, -f(z) \rangle_{\mathbb{L}(\psi)} .$$

Therefore $f(z) \equiv 0$, which means that $\mathbb{L}$ is the null space. For the converse, we assume that $\mathbb{L} = \{0\}$. If the minimal decomposition of some $f(z)$ belonging to $\mathbb{L}(\theta)$ be $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is in $\mathbb{L}(\theta)$ and $f_2(z)$ is in $\mathbb{L}(\psi)$, then $f_2(z) = f(z) - f_1(z)$ belongs to $\mathbb{L}$ which has no non-zero element. Therefore $f(z) = f_1(z)$ and $\| f(z) \|_{\mathbb{L}(\phi)} = \| f(z) \|_{\mathbb{L}(\theta)}$. It implies that $\mathbb{L}(\theta)$ is contained in $\mathbb{L}(\phi)$ isometrically.

Let $\mathbb{L}(\chi)$ be the space whose extension is $\mathbb{E}(\chi)$. Let $h(z) = f(z) + g(z)$ be the minimal decomposition of some $h(z)$ in $\mathbb{L}(\phi)$ with $f(z)$ in $\mathbb{L}(\theta)$ and $g(z)$ in $\mathbb{L}(\psi)$. If $p(z)$ belongs to $\mathbb{L}(\chi)$ then by (5.1.2), we have

$$\langle h(z), 0 \rangle_{\mathbb{L}(\phi)} = \langle f(z), p(z) \rangle_{\mathbb{L}(\theta)} - \langle g(z), p(z) \rangle_{\mathbb{L}(\psi)} . \tag{5.1.3}$$

If $h(z)$ is a polynomial in $\mathbb{L}(\theta)$, $\| h(z) \|_{\mathbb{L}(\phi)} = \| h(z) \|_{\mathbb{L}(\theta)}$ by hypothesis, and the minimal decomposition is obtained with $f(z) = h(z)$ and $g(z) = 0$ in (5.1.3). So for every $p(z)$ in $\mathbb{L}(\psi)$

$$\langle h(z), p(z) \rangle_{\mathbb{L}(\phi)} = \langle f(z), p(z) \rangle_{\mathbb{L}(\theta)} = \langle g(z), p(z) \rangle_{\mathbb{L}(\psi)} = 0 .$$

By the arbitrariness of $p(z)$ in $\mathbb{L}(\chi)$, it follows that $\mathbb{L}(\chi)$ has no non-zero polynomial. Therefore, it implies that

$$\| f(z) \| \ldots = \left\| \frac{f(z) - f(0)}{\ldots} \right\|_{\mathbb{L}(\chi)}$$

for all $f(z)$ of $\mathbb{L}(\chi)$, which means that $f(z) \to \dfrac{f(z)-f(0)}{z}$ is a unitary transformation in $\mathbb{L}(\chi)$. This completes the proof of the Lemma. The following Lemma is an application of Lemma 1.

LEMMA 2. Let $H(B)$ be a given space such that the elements of the form $B(z)l(z)$, $l(z)$ in $\mathbb{L}(\phi)$ the overlapping space of $H(B)$, form a dense set in $H(B)$. Then $H(B)$ is the null space, if and only if, there is no non-zero space $\mathbb{L}(\psi)$ contained in $\mathbb{L}(\phi)$ and $\mathbb{L}(1-\phi)$ such that the inclusions do not increase norms.

Proof: One way is trivially true. Assume that the given space $H(B)$ is non-null, then there exists an element $l(z)$ in $\mathbb{L}(\phi)$ such that $\| l(z)\|_{\mathbb{L}(\phi)} > \| l(z)\|$, which implies that the space $\mathbb{L}(\phi)$ is not contained isometrically in $\mathbb{L}(1)$. Therefore by Lemma 1, $\mathbb{L}(=\mathbb{L}(\phi)\cap\mathbb{L}(1-\phi))$ is a non-null space and is equal to $\mathbb{L}(\psi)$ for some $\psi(z)$. By definition of the norm in $\mathbb{L}(\psi)$, the inclusions of $\mathbb{L}(\psi)$ in $\mathbb{L}(\phi)$ and $\mathbb{L}(1-\phi)$ do not increase norms. Hence the Lemma.

The lemma can be used to characterize $B(z)$ where $H(B)$ is given by Theorem 5 ([7] ; p. 126).

COROLLARY 1. Let $H(B)$ be a given space such that the set of elements of the form $B(z)l(z)$, $l(z)$ in $\mathbb{L}(\phi)$ the overlapping space of $H(B)$, is dense in $H(B)$. Then the multiplication by $B^*(z)$ is not isometric unless $H(B)$ is the null space.

Proof: If the multiplication by $B^*(z)$ is $S(z)$ is isometric, then by Theorem 5 ([7]; p. 126), there is no non-zero space $\mathbb{L}(\psi)$ contained in $\mathbb{L}(\phi)$ and $\mathbb{L}(1-\phi)$ such that the inclusions do not

increase norms. Therefore $\mathbb{L}(\emptyset)$ is contained isometrically in $\mathbb{L}(1)$, which means $B(z)l(z)=0$ for every element $l(z)$ of $\mathbb{L}(\emptyset)$. Hence $H(B)$ is null.

Following Theorem illustrates an use of $H(B)$ space theory in invariant subspaces theory.

THEOREM 1: Let T be a contraction (i.e. a transformation bounded by 1) in a Hilbert space H of dimension greater than 1. Assume that either $T^n \not\to 0$ or $T^{*n} \not\to 0$ as $n \to \infty$ in the norm of H. Then there exists a non-zero closed proper subspace of H, invariant under T.

Proof: If H is unseparable space, then for every non-zero element f in H, the closed span of $T^n f$, $n=0,1,2\ldots$, is a non-trivial subspace, invariant under T. So we assume that H is separable. By a result from [24], it follows that T can be decomposed uniquely as a direct sum of a unitary transformation $T^{(u)}$ and a completely non-unitary transformation $T^{(c)}$ (T is said to be c.n.u if for every non-zero element f in H, there exists positive integer $n=n(f)$ such that $\|T^n f\| < \|f\|$ or $\|T^{*n} f\| < \|f\|$). If $T^{(c)}$ is zero then T is unitary on H and therefore its invariant subspaces can be obtained by knowing its spectral integral representation. If $T^{(c)}$ is non-zero then the domain of $T^{(u)}$ is a closed, proper subspace of H invariant under T. So in what follows we shall assume that there is no non-zero subspace M in H such that the restriction $T/_M$ is unitary in M. Consider the closed space N of elements f of H such that $\|T^{*n} f\| = \|f\|$, $n=1,2,\ldots$ The orthogonal complement of N in H is the required invariant subspace of T if it is non-null and properly contained in H. If $N = H$ then T is a partial

isometry which contradicts our assumption, so we assume that $N=\{0\}$. Following similar arguments, we assume that there is non-zero f in H such that $\|T^n f\| = \|f\|$ for all $n \geq 1$, otherwise T has a non-trivial invariant subspace. Thus in what follows we assume that for every non-zero f in H $\|T^n f\| < \|f\|$ and $\|T^{*n} f\| < \|f\|$ for some $n > 0$.

Because H is separable, the dimension of the closure of the range of $(1-T^*T)$ is countable. Let S be a infinite dimensional (separable) Hilbert space. By the results of the section (1.2.11), T is unitarily equivalent to the transformation

$$R(0): f(z) \to \frac{f(z)-f(0)}{z}$$

in a space H(B), where B(z) is some series with operator coefficients such that the identity

$$(5.1.4) \qquad \left\| \frac{f(z)-f(0)}{z} \right\|_B^2 = \|f(z)\|_B^2 - |f(0)|^2$$

holds for every f(z) in H(B). The transformation $U: f(z) \to \tilde{f}(z)$ (defined by (1.2.6)) on H(B) is a partial isometry onto $H(B^*)$.
If $U f(z)=0$ for some f(z) in H(B) then $R(0)^{*n} f(z)=z^n f(z)$ which means that $\|R(0)^{*n} f(z)\|_B = \|f(z)\|_B$, for all $n=0,1,2,\ldots$ It follows that $f(z)=0$. So U is an isometric transformation defined on H(B) onto $H(B^*)$. We now assume that $R(0)^n f(z) \not\to 0$ as $n \to \infty$ for some f(z) is H(B), which means that there exists an element f(z) in H(B) such that $\|f(z)\|_B > \|f(z)\|$. So H(B) is not contained isometrically in $S(z)$.

Consider the closed subspace of $H(B)$ which is isometrically contained in $S(z)$. This contains $\frac{f(z)-f(0)}{z}$ (because of (5.1.4)) whenever it contains $f(z)$, so it is the required invariant subspace, provided there exists a non-zero $f(z)$ such that $\|f(z)\|_B = \|f(z)\|$. So in what follows we assume that $\|f(z)\|_B > \|f(z)\|$ for all $f(z)$ ($\neq 0$) of $H(B)$, which in other words means that the orthogonal complement of the range of multiplication by $B(z)$ in $S(z)$ is null. So the elements of the form $B(z)l(z)$ with $l(z)$ in $\mathbb{L}(\phi)$ the overlapping space of $H(B)$, form a dense set in $H(B)$. Therefore by Lemma 2, there exists a non-zero space $\mathbb{L}(\dot{\psi})$ contained in $\mathbb{L}(\phi)$ and $\mathbb{L}(1-\phi)$ such that the inclusions do not increase norms. We also note that there is no non-zero $B(z)c$ in $H(B)$. Since $\frac{l(z)-l(0)}{z}$ belongs to $\mathbb{L}(\phi)$ along with $l(z)$, hence $B(z)l(z)=0$ whenever $l(z)$ is a polynomial in $\mathbb{L}(\phi)$. So $\|l(z)\|_{(\phi)} = \|l(z)\|$ for all polynomial $l(z)$ of $\mathbb{L}(\phi)$. At this stage we can use Theorem 3 and 4 ([7]; p. 126) to show that $B(z)=A(z)C(z)$ such that $H(A)$ is a non-null proper subspace of $H(B)$. But we give here an easier argument.

Consider the closed span of elements $l(z)$ of $\mathbb{L}(\phi)$ such that $\|l(z)\|_{\mathbb{L}(\phi)} = \|l(z)\|$. Obviously, it is not whole of $\mathbb{L}(\phi)$. If $l(z)$ belongs to the span, then $B(z)l(z)=0$ and so if

(5.1.5) $$\bar{c}\,\tilde{l}(0) = \langle l(z), \tfrac{1}{2}\,\frac{\phi(z)-\phi(0)}{z}c\rangle_{\mathbb{L}(\phi)}$$

then

$$\|zl(z)+\tilde{l}(0)\|^2_{\mathbb{L}(\phi)} = \|zB(z)l(z)+B(z)\tilde{l}(0)\|^2_B + \|zl(z)+\tilde{l}(0)\|^2$$

$$= \|zl(z)+\tilde{l}(0)\|^2$$

because H(B) has no non-zero element of the form $B(z)c$ for any constant c. It implies that the closed span is invariant under the adjoint of $l(z) \to \frac{l(z)-l(z)}{z}$ defined in $\mathbb{L}(\phi)$. Therefore its orthogonal complement is a closed subspace of $\mathbb{L}(\phi)$, which contains $\frac{l(z)-l(0)}{z}$ whenever $l(z)$ belongs to it, and

$$(5.1.6\ ) \qquad \left\| \frac{l(z)-l(0)}{z} \right\|_{\mathbb{L}(\phi)} = \| l(z) \|_{\mathbb{L}(\phi)} .$$

Therefore the complement is equal to some $\mathbb{L}(\theta)$, which is obviously a non-null space. Infact, it is of dimension greater than 1. The multiplication by $B(z)$ on $\mathbb{L}(\theta)$ is a one-to-one transformation and has dense range in H(B). Since by (5.1.6), the transformation is unitary, so $\mathbb{L}(\theta)$ has a non-trivial subspace M invariant under $l(z) \to \frac{l(z)-l(0)}{z}$. We claim that the image of M in H(B) under the multiplication transformation by $B(z)$, is invariant under the adjoint of $f(z) \to \frac{f(z)-f(0)}{z}$. Let $B(z)l(z)$ be in the image of M for some $l(z)$ in M. Since $l(z) \to zl(z)+\tilde{l}(0)$ ($\tilde{l}(0)$ is the same as defined in (5.1.5)) under the adjoint of $l(z) \to \frac{l(z)-l(0)}{z}$, hence $zB(z)l(z)+B(z)\tilde{l}(0)$ belong to the image of M. It follows from Theorem 11 ([6] p. 349) that

$$R(0)^*:\ B(z)l(z) \to \quad zB(z)l(z)+B(z)\tilde{l}(0).$$

This proves that the closure of the image of M in H(B) is a non-null proper subspace of H(B), which contains $R(0)^*f(z)$ whenever it contains $f(z)$. Its orthogonal complement is the required invariant subspace of $R(0)$.

In the rest of the proof we assume that $R(0)^n \to 0$ in the norm of $H(B)$ but there exists an element $g(z)$ such that $R(0)^{*n}g(z) \not\to 0$ as $n \to \infty$. In other words, it means that the multiplication by $B(z)$ is isometric in $S(z)$ whereas the multiplication by $B^*(z)$ is not partially isometric. So the closure of the elements of the form $B^*(z)l(z)$ with $l(z)$ in $S(z)$, is a non-null subspace of $H(B^*)$. With the previous arguments, one can show that this closure is invariant under the adjoint of the transformation $R_1(0): f(z) \to \frac{f(z)-f(0)}{z}$ in $H(B^*)$. Since $H(B^*)$ is isometrically isomorpnic to $H(B\ )$ under the transformation $V: f(z) \to \hat{f}(z)$, where

$$\bar{c}\ \hat{f}(w) = \langle f(z), \frac{B^*(z)-\bar{B}(w)}{z-\bar{w}}\, c \rangle_{B^*}$$

and $VR_1(0)^* = R(0)V$, the image of the closure of elements of the form $B^*(z)l(z)$ in $H(B^*)$ with $l(z)$ in $S(z)$, is a non-null closed subspace of $H(B)$ which contains $\frac{f(z)-f(0)}{z}$ whenever it contains $f(z)$, It remains to show that it is properly contained in $H(B)$. If it coincides with $H(B)$, then the set of elements of the form $B^*(z)l(z)$ in $H(B^*)$ is dense in $H(B^*)$. By the corollary of Lemma 2, it then follows that $H(B^*)$ is the null space which is not possible. This completes the proof of the Theorem.

If we summarize the proof of the Theorem, we get the following Corollary.

<u>COROLLARY:</u> Let T be a contraction on an infinite dimensional (separable) Hilbert space H. Assume that there is no non-zero element

f in H such that $\| T^n f \| = \|f\|$ for all $n \geq 1$. If T^{*n} converges to zero strongly, then there exists a non-zero element g such that $\| T^n g \| \to 0$ as $n \to \infty$. Hence the closed span of such elements g, is a non-null invariant subspace of T.

5.2 The following Theorem shows that $B(z)$ can be factorized if either of operators $1-B(0)\overline{B}(0)$ or $1-\overline{B}(0)B(0)$ is completely continuous.

THEOREM 2: Let T be a contraction on a Hilbert space H into itself. Assume that T is bounded by 1 and either of the transformations $(1-T^*T)$ or $(1-TT^*)$ is completely continuous. If H is of dimension greater than 1 then T has a non-trivial invariant subspace.

We require following Lemmas to prove the Theorem. Throughout the chapter the coefficient space S is assumed to be separable.

LEMMA 3: A necessary and sufficient condition that $H(B)$ is finite dimensional is that

$$B(z)=B_1(z)B_2(z)\ldots\ldots B_n(z), \tag{5.2.1}$$

where

$$2\,(I+B_k(z))^{-1} = I + \frac{1+z\overline{\lambda}_k}{1-z\lambda_k}\; c_k\overline{c}_k \,, \tag{5.2.2}$$

the (c_k) are vectors and $|\lambda_k| = 1$. The factors can be so chosen that $H(B_1B_2\ldots B_k)$ is contained isometrically in $H(B)$ for all $k=1,2,\ldots,n$.

LEMMA 4: Let $H(B)$ be a given space such that the operator $1-B(0)\overline{B}(0)$ is completely continuous. Then there exists a sequence of finite

dimensional space $H(B_n)$ such that $B(w) = \lim_{n\to\infty} B_n(w)$ for all complex number w, $|w|<1$.

Following Lemma extends the result of Lemma 6 ([5]; p. 53).

LEMMA 5: If $H(B_n)$ is any sequence of spaces such that $1-B_n(0)\bar{B}_n(0)$ is completely continuous operator for $n=1,2,\ldots$, then there exists a subsequence $\{n_k\}$ of positive integer such that

$$\bar{c}\,\frac{1-B(\alpha)\bar{B}(w)}{1-\alpha\bar{w}}\,c = \lim_{k\to\infty} \bar{c}\,\frac{1-B_{n_k}(\alpha)\,\bar{B}_{n_k}(w)}{1-\alpha\bar{w}}\,c$$

for each constant c and complex numbers $|\alpha|<1$, $|w|<1$.

LEMMA 6: Let $H(B)$ be a given space such that the multiplications by $B(z)$ is isometric in $S(z)$ and $1-B(0)\bar{B}(0)$ is completely continuous. Then $B(z)=A(z)P(z)C(z)$ such that $H(P)$ is 0 or 1 dimensional and $H(A)$ is isometrically contained in $H(B)$.

Proof [redacted]: Let the given space $H(B)$ is of dimension 1. If

$$\emptyset(z) = \frac{1-B(z)}{1+B(z)} \tag{5.2.3}$$

then $\mathbb{L}(\emptyset)$ is also 1 dimensional. Since $\mathbb{L}(\emptyset)$ contains $\frac{1(z)-1(0)}{z}$

(i) If a and b are vectors then $a\bar{b}$ denotes the operator defined by $a\bar{b}(c) = a(\bar{b}c)$ for all vectors c.

(ii) The results of Theorem 2, Lemma 5 and 6 were announced in ([4]; p.396) and ([4]; p. 720) respectively, but the author failed to locate the proofs of these announcements.

whenever it has $l(z)$, $\frac{c}{1-\bar{\lambda}z}$ spans $\mathbb{L}(\phi)$, where c is some constant and $|\lambda| \leq 1$. As $\mathbb{L}(\phi)$ can not have any non-zero polynomial, so

$$\left\| \frac{l(z)-l(0)}{z} \right\|_{\mathbb{L}(\phi)} = \| l(z) \|_{\mathbb{L}(\phi)}$$

for every $l(z)$ in $\mathbb{L}(\phi)$. It implies that $|\lambda| = 1$. By the definition of $k(w,z)$, we get,

$$k(w,z) = \frac{1}{2} \quad \frac{\phi(z)+\bar{\phi}(w)}{1-z\bar{w}} = \frac{c\bar{c}}{(1-\bar{w}\lambda)(1-z\bar{\lambda})} ,$$

or ,

$$\frac{1}{2} \ (\phi(z) + \bar{\phi}(w)) = \left(1 + \frac{\bar{w}\lambda}{1-\bar{w}\lambda} + \frac{z\bar{\lambda}}{1-z\bar{\lambda}} \right) c\bar{c} .$$

Therefore,

$$1+\phi(z) = 1 + \left(\frac{1+z\bar{\lambda}}{1-z\bar{\lambda}} \right) c\bar{c} . \tag{5.2.4}$$

Combining (5.2.3) and (5.2.4), we get,

$$2(1+B(z))^{-1} = 1 + \left(\frac{1+z\bar{\lambda}}{1-z\bar{\lambda}} \right) c\bar{c} .$$

Now let $H(B)$ be of finite dimension (say n). Then $R(0)$ has an invariant subspace of dimension $(n-1)$. So $B(z)=A(z)C(z)$, where $H(A)$ is 1-dimensional. Continuing the same arguement, (5.2.1) follows by induction. The last part of the Lemma follows from the fact that $H(A)$ is isometrically contained in $H(B)$.

Proof of Lemma 4: Assume that $(1-B^{*}(0)\bar{B}(0)$ is completely continuous operator. If $\phi(z)$ is defined by (5.2.3), then Re $\phi(0)$ is completely

continuous. The function $\phi(z)$ can be so chosen without changing the space $\mathbb{L}(\phi)$ such that

$$\phi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \, d\,\mu(\theta),$$

in the unit disc, where $\mu(\theta)$ is a non-decreasing operator-valued measure. By Lemma 3 ([26] ; p. 58), it follows that there exists a sequence of spaces $\mathbb{L}(\phi_n)$ for n=1,2,..., such that the dimension of $\mathbb{L}(\phi_n)$ is n and

$$\phi(z) = \lim_{n\to\infty} \phi_n(z)$$

formally (coefficient wise convergence). Define

$$B_n(z) = \frac{1-\phi_n(z)}{1+\phi_n(z)}$$

then each $H(B_n)$ is a finite dimensional space. Moreover, $B(z) = \lim_{n\to\infty} B_n(z)$ formally, which implies that

$$B(w) = \lim_{n\to\infty} B_n(w)$$

for all complex number w, $|w| < 1$. Hence the Lemma.

<u>Proof of Lemma 5</u>: The Lemma can be proved on the lines, that used in the proof of Lemma 6 ([5]; p. 53).

<u>Proof of Lemma 6</u>: By Lemma 4, there exists a sequence of finite dimensional space $H(B_n)$ such that $B(z) = \lim_{n\to\infty} B_n(z)$, and $1-B_n(0)\overline{B}_n(0)$ is completely continuous for n=1,2,.... Let c be a vector and h_n be numbers such that

(5.2.5) $$0 \le h_n \le |c|^2 - |\overline{B}_n(0)\, c|^2 .$$

for every n. Let $h = \lim_{n\to\infty} h_n$. Since $H(B_n)$ is of dimension n,

$$B_n(z) = B_{n_1}(z)\cdot\ldots\cdot B_{n_n}(z)$$

and $$1-B_{n_k}(0)\,\overline{B}_{n_k}(0) = c_{n_k}\overline{c}_{n_k}$$

So, $B_n(z) = D_n(z)C_n(z)$, such that

$$h_n \leq |c|^2 - |D_n(0)c|^2,$$

and $H(D_n)$ is isometrically contained in $H(B_n)$. Applying the same argument it follows that $D_n(z)=A_n(z)P_n(z)$ such that

$$|c|^2 - |A_n(0)\,c|^2 \leq h_n ,$$

$H(P_n)$ is of dimension 0 or 1 and $H(A_n)$ is isometrically contained in $H(D_n)$. By Lemma 5, there exists a subsequence n_k of positive integer such that $A(z) = \lim A_{n_k}(z)$, $P(z) = \lim P_{n_k}(z)$ and $C(z)=\lim C_{n_k}(z)$. By the construction, it follows that $H(A)$ is contained in $H(B)$. Since $H(B)$ is contained isometrically in $\mathcal{S}(z)$, the multiplication by $A(z)$ and $C(z)$ are partially isometric transformations. So the inclusion of $H(A)$ in $H(B)$ is isometric. Since each $H(P_n)$ is of dimension 0 or 1, so $H(P)$ is also of dimension 0 or 1. This proves the Lemma.

<u>Proof of Theorem 2:</u> We assume that T^n and T^{*n} both converge to zero strongly as $n\to\infty$, otherwise by Theorem 1, T has non-trivial invariant subspaces

We can also assume(by the proof of Theorem 1)that $H=H(B)$ is an infinite dimensional (separable)space, $T=R(0)$, the coefficient space S is separable and the multiplications by $B(z)$ and $B^*(z)$ are isometric transformations in $S(z)$. We assume that $1-T^*T$ is completely continuous which implies that $1-B(0)\bar{B}(0)$ is a completely continuous operator. By Lemma 6, $B(z)=A(z)F(z)C(z)$ such that the spaces $H(A)$ and $H(AF)$ are contained isometrically in $H(B)$. By construction $H(AP)$ is a proper subspace of $H(B)$ and is the required invariant subspace of T.

If $(1-TT^*)$ is completely continuous then the same arguments can be given for $H(B^*)$ to get a similar factorization of $B^*(z)$. So in that case too, T has a non-trivial invariant subspace. This completes the proof of the Theorem.

5.4 The following section concerns with the existence of invariant subspaces of a contraction transformation T such that $T^n \to 0$ and $T^{*n} \to 0$ as $n \to \infty$. In the view of the proof of Theorem 1, we can assume that the given space is $H(B)$ for some $B(z)$ with operator coefficients and the given transformation is $R(0)$: $f(z) \to \dfrac{f(z)-f(0)}{z}$ in $H(B)$ such that $H(B)$ and $H(B^*)$ are isometrically contained in $S(z)$, i.e. the multiplications by $B(z)$ and $B^*(z)$ in $S(z)$ are partially isometric transformations. (We also assume that $H(B)$ and $H(B^*)$ both are infinite dimensional spaces). We determine the invariant subspaces of $R(0)$ which can be obtained by factorizing $B(z)$, in case $B(z)$ is a polynomial.

Since $H(B)$ is contained isometrically in $S(z)$, the closed subspace of constants which belong to $H(B)$, is equal to $H(A_0)$, where

$A_0(z) = P_0 + P_0' z$. It is obtained by an easy computation that $P_0' = 1-P_0$, P_0 is an projection on S and the range of P_0' in S is isometrically equal to $H(A_0)$. Therefore a space $H(C_0)$ exists such that $B(z)=A_0(z)C_0(z)$ and the multiplication by $C_0(z)$ is an isometry in $S(z)$. Continuing inductively, we obtain that $B(z)=A_n(z)C_n(z)$, where

$$A_n(z) = (P_0 + (1-P_0)z)(P_1 + (1-P_1)z) \ldots (P_n + (1-P_n)z).$$

Since $B(z)$ is a polynomial, and $R(0)^n \to 0$ as $n \to \infty$, the closed span of

$$\{ R(0)^n \frac{B(z)-B(0)}{z} ; n=0,1,2\ldots, c \text{ in } S \}$$

is whole of $H(B)$. Therefore the process of induction terminates at some n (depending on the degree of $B(z)$). Therefore, it follows that $H(C_n)$ has no non-zero element for some n, which means that $C_n(z)=C_n(0)$ is an unitary transformation. Thus $B(z)$ can be written as

$$B(z) = (P_0 + (1-P_0)z) \ldots (P_n + (1-P_n)z) C_n(0),$$

where P_i, $i=0,1,\ldots,n$ are projections on S.

Assume that $B(z)$ is an infinite power series. If $H = H(B) \cap H(B^*)$ then H is a closed subspace of $H(B)$ which contains $\frac{f(z)-f(0)}{z}$ whenever it contains $f(z)$ and is the required invariant subspace, if it is a non-trivial subspace. So it is assumed that either $H = \{0\}$ or $H = H(B)$. If $H(B) = H(B^*)$, then by inclusion theory, $B^*(z) = UB(z)$,

where U is an unitary transformation on S. Since $B(z)$ is bounded by 1 in the unit disc and assumes unitary values almost everywhere on the unit circle, hence $\int_0^{2\pi} \|B(e^{i\theta})\| \, d\theta < \infty$.

Therefore by Theorem 4 ([21]), $B(z)=VA^*(z)A(z)$, where $A(z)$ is some power series with operator coefficients which converses in the unit disc and V is some unitary transformation S. Since $B(z)$ is bounded by 1, so $A(z)$ is also bounded by 1 in the unit disc. It also follows that the multiplications by $A(z)$ and $A^*(z)$, are isometric in $S(z)$. It is easy to observe that $H(A^*)$ is non-null, otherwise $A(z)=A(0)$ and so $B(z)=B(0)$. It also says that $H(B) \neq H(A^*)$, because then $A^*(z)\ (A(z)-A(w))=0$ for each complex number w, $|w| < 1$, which again implies that $A(z)=A(0)$. So $H(A^*)$ is contained isometrically in $H(B)$ and is a non-null proper subspace of $H(B)$, which contains $\frac{f(z)-f(0)}{z}$ alongwith $f(z)$. So $R(0)$ has a non-trivial invariant subspace if $H(B) = H(B^*)$. So in what follows we assume that $H(B) \cap H(B^*) = \{0\}$ i.e. $M(B) + M(B^*) = S(z)$, where $M(B)$ and $M(B^*)$ are the ranges of multiplications by $B(z)$ and $B^*(z)$ in $S(z)$, respectively. If $M(B) \cap H(B^*) \neq \{0\}$ then $H(B^*)$ contains a non-zero element of the form $B(z)f(z)$ where $f(z)$ is some element in $S(z)$. Since the multiplications by $B(z)$ and $B^*(z)$ are isometrics in $S(z)$, an easy calculation shows that $B(z)\ \frac{f(z)-f(0)}{z}$ belongs to $H(B^*)$. But as $\frac{B(z)f(z)-B(0)f(0)}{z}$ belongs to $H(B^*)$ so $\frac{B(z)-B\ 0)}{z}\ f(0)$ belongs to $H(B^*)$, which either contradicts the assumption that $H(B) \cap H(B^*) = \{0\}$, or implies that $\frac{B(z)-B(0)}{z}\ f(0)=0$.

If $\frac{B(z)-B(0}{z}\ f(0)=0$ then $f(0)$ is orthogonal to the coefficients of elements of $H(B^*)$. In that case one can prove that $H(B^*)$ has a non-trivial invariant subspace of $f(z) \rightarrow \frac{f(z)-f(0)}{z}$ and hence $H(B)$ has a non-trivial invariant subspace of $R(0)$.

By summing we get that a space $H(B)$ contained isometrically in $\mathcal{S}(z)$ such that $H(B^*)$ is also contained isometrically in $\mathcal{S}(z)$, has a non-trivial closed subspace which contains $\frac{f(z)-f(0)}{z}$ whenever it contains $f(z)$, if either $H(B)\cap H(B^*) \neq \{0\}$ or, $\mathcal{M}(B)\cap H(B^*) \neq \{0\}$.

The spectrum of $R(0)$ in $H(B)$ is directly connected with the existence of invariant subspaces of $R(0)$. A point w in the unit disc, is an eigenvalue of $R(0)$, if and only if, $\frac{c}{1-z\overline{w}}$ belongs to $H(B)$ for some non-zero constant c, which is equivalent to say that $\frac{B(z)\overline{B}(w)}{1-z\overline{w}}c$ belongs to $H(B)$ which is possible, if and only if, $\overline{B}(w)c=0$, i.e. when $B(w)$ fails to have dense range in $\mathcal{S}$. If $(R(0)-\overline{w})$ fails to have dense range in $\mathcal{S}$ for some number w, $|w|<1$, i.e. if w is an eigenvalue of $R(0)^*$ then it can be shown that $B(w)c=0$ for some non-zero constant c, implying thereby that $B(w)$ is not 1-1. Hence the problem of testing the existence of invariant subspaces is equivalent to factorize operator-valued function $B(z)$ which has infinite power series expansion such that (i) the multiplications by $B(z)$ and $B^*(z)$ are isometries in $\mathcal{S}(z)$ and (ii) $\overline{B}(w)$ and $B(w)$ have dense range in $\mathcal{S}$ for every complex number w, $|w|<1$.

Appendix on $\mathbb{L}(\phi)$ Spaces

The following is a simple application of $\mathbb{L}(\phi)$ spaces theory, which gives a significant result.

THEOREM 1. Let $\phi(z) = \sum_{n=0}^{\infty} A_n z^n$ be an operator-valued analytic function in the unit disc such that

$$0 \leq \mathrm{Re}\, \phi(z) = \frac{1}{2} (\phi(z)+\bar{\phi}(z)) \leq I$$

for every z in the unit disc. Then

$$\sum_{n=1}^{\infty} \bar{A}_n A_n \leq 4 \,\mathrm{Re}\, A_o (I - \mathrm{Re}\, A_o) .$$

Moreover, if $\mathrm{Re}\, \phi(w) = 0$ for some w, $|w| < 1$, then $\phi(z) = iA$, where A is some self-adjoint operator.

Proof. By the given hypothesis and the proof of Theorem 1 ([7] ; p. 126), the spaces $\mathbb{L}(\phi)$ and $\mathbb{L}(I-\phi)$ exist, and are contained in $S(z)$ such that their inclusions do not increase norms. Therefore, $\frac{1}{2} (\phi(z)+\bar{\phi}(0))c$ belongs to $S(z)$ for every constant c, and

$$\left\| \frac{1}{2} (\phi(z)+\bar{\phi}(0))c \right\|^2 \leq \left\| \frac{1}{2} (\phi(z)+\bar{\phi}(0))c \right\|^2_{\mathbb{L}(\phi)} = \frac{1}{2} \bar{c}(\phi(0)+\bar{\phi}(0))c,$$

or equivalently,

$$\frac{1}{4} \sum_{n=1}^{\infty} |A_n c|^2 + |(\mathrm{Re}\, A_o)c|^2 \leq \bar{c}\, (\mathrm{Re}\, A_o)c.$$

By the arbitrariness of c,

$$\sum_{n=1}^{\infty} \overline{A}_n A_n \leq 4 \operatorname{Re} A_0 (I - \operatorname{Re} A_0) .$$

Again, if Re $\phi(w) = 0$ for some number w, $|w| < 1$, then

$$\left\| \frac{1}{2} \frac{\phi(z)+\overline{\phi}(w)}{1-z\overline{w}} c \right\|^2 \leq \left\| \frac{1}{2} \frac{\phi(z)+\overline{\phi}(w)}{1-z\overline{w}} c \right\|^2_{\mathbb{L}(\phi)}$$

$$= \overline{c} \operatorname{Re} \phi(w) c .$$

$$= 0,$$

i.e., $\quad (\phi(z)+\overline{\phi}(w))c = 0$

for every constant c. Hence $\phi(z) = -\overline{\phi}(w) = iA$, for some self adjoint operator A.

THEOREM 2. Let $\mathbb{L}(\phi)$ be a given space. The transformation $l(z) \to \tilde{l}(z)$ defined on $\mathbb{L}(\phi)$ into $\mathbb{L}(\phi^*)$ is isometric, if and only if, $\mathbb{L}(\phi^*)$ has no non-zero constant, where

$$\overline{c} \; \tilde{l}(w) = \left\langle l(z), \frac{1}{2} \frac{\phi(z)-\overline{\phi}(w)}{z-\overline{w}} c \right\rangle_{\mathbb{L}(\phi)}$$

for every number w, $|w| < 1$ and constant c.

Proof. Let $\mathbb{E}(\phi)$ and $\mathbb{E}(\phi^*)$ be the extension spaces of $\mathbb{L}(\phi)$ and $\mathbb{L}(\phi^*)$, respectively. If $\mathbb{L}(\phi^*)$ has no non-zero constant, then there is no element $(f(z),g(z))$ in $\mathbb{E}(\phi)$ such that $g(z)=c$ for any non-zero constant c. Since $\left(\frac{f(z)-f(0)}{z}, zg(z) + f(0)\right)$ belongs to $\mathbb{E}(\phi)$ whenever $(f(z),g(z))$ belongs to $\mathbb{E}(\phi)$, $\mathbb{E}(\phi)$ contains no non-zero element $(f(z),g(z))$ such that $g(z) = 0$. By the definition of $\mathbb{E}(\phi^*)$,

$$\| (f(z),g(z)) \|_{\mathbb{E}(\phi)} = \| g(z) \|_{\mathbb{L}(\phi^*)} .$$

Since the transformation $l(z) \rightarrow (l(z), \tilde{l}(z))$ is an isometry on $\mathbb{L}(\phi)$ into $\mathbb{E}(\phi)$, so $l(z) \rightarrow \tilde{l}(z)$ is an isometric transformation on $\mathbb{L}(\phi)$ into $\mathbb{L}(\phi^*)$.

Conversely, let $l(z) \rightarrow \tilde{l}(z)$ be isometric. Then the range of the transformation $l(z) \rightarrow (l(z), \tilde{l}(z))$ in $\mathbb{E}(\phi)$ is orthogonal to elements of the form $(f(z),g(z))$ such that $g(z)=0$. But as the orthogonal complement of the range consists only of elements of the form $(f(z),g(z))$ where $f(z)=0$, hence there is no non-zero element $(f(z),g(z))$ in $\mathbb{E}(\phi)$ such that $g(z) = 0$. If c is an element in $\mathbb{L}(\phi^*)$ then there exists an element $f(z)$ in $\mathbb{L}(\phi)$ such that $(f(z),c)$ belongs to $\mathbb{E}(\phi)$, i.e., $(c,f(z))$ belongs to $\mathbb{E}(\phi^*)$. Therefore $(0,zf(z)+c)$ belongs to $\mathbb{E}(\phi^*)$. As $(zf(z)+c,0)$ belongs to $\mathbb{E}(\phi)$ it implies that $c=0$. Hence the Theorem.

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